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On fibrous slender materials

Cz. WOŹNIAK (WARSZAWA)

THE AIM of the paper is to describe and discuss materials made of smooth and intersecting slender fibres, i.e. fibres which are not able to sustain compressive forces. It is shown that, due to the slenderness of fibres, the related problems lead to certain systems of quasi-variational inequalities. The obtained relations can be applied to an analysis of some problems concerning fabrics and other textile-like materials.

Celem pracy jest opis i dyskusja materiałów utworzonych z gładkich i przecinających się wzajemnie wiotkich włókien, tj. włókien nie przenoszących sił ściskających. Wykazano, że założenie wiotkości włókien prowadzi do zagadnień opisywanych układami nierówności quasi-wariacyjnych. Otrzymane relacje mogą być stosowane do analizy niektórych problemów dotyczących materiałów tekstylnych.

Целью работы является описание и обсуждение материалов, образованных из гладких и пересекающихся взаимно гибких волокон, т.е. волокон не передающих сжимающих сил. Показано, что предположение гибкости волокон приводит к задачам описыванным системами квазивариационных неравенств. Полученные зависимости могут быть применены к анализу некоторых задач, касающихся текстильных материалов.

1. Introduction

THE OBJECTIVE of the paper is to formulate and discuss governing relations of materials made of a few families of mutually intersecting smooth fibres. The constitutive modelling of materials under consideration will be based on the following assumptions:

1. Every fibre is slender, i.e. it is unable to sustain any compressive force, bending couple and torque.

2. Every configuration of a material is represented by m, $m \geqslant 2$, virtually continuous families of smooth lines which occupy a regular region on a plane or on a smooth surface.

3. The material response depends exclusively on the material properties of fibres and on the geometric structure of a fibre lattice.

The ideal materials under consideration will be referred to as the fibrous slender materials; in a separate paper we are to show that they can be applied to the analysis of some special problems concerning fabrics and other textile-like materials. The approach presented here is different from the one given in [1-3] where the "slenderness" of a material was not related to its fibrous structure.

2. Modelling of elastic-slender deformations

Let Ω be the regular plane region and $t_A: \bar{\Omega} \to R^2$, $A = 1, \ldots, m$, $m \geqslant 2$, be continuous fields of unit vectors such that $|t_A(X) \cdot t_B(X)| < 1$ for every $A \neq B$ and every $X \in \bar{\Omega}$.

Following the approach used in [4] for the modelling of smooth, thin and regularly distributed families of cords, we shall assume that the vectors $t_A(X)$ are tangent to the A-th family of fibres, $A = 1, \ldots, m$ (cf. Fig. 1 where $m = 3$). Thus we introduce the virtually continuous description of the fibre lattice under consideration. If $m = 3$, then we deal with the plane lattice in the sense of W. BLASCHKE, [5]. Let $\mathscr{D}$ be a set of smooth invertible

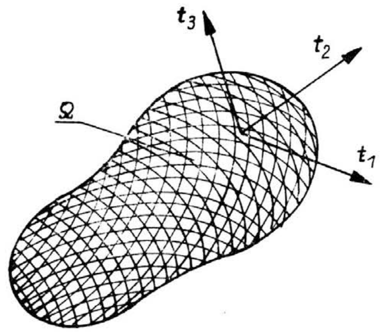

FIG. 1.

mappings (deformations) $p\colon \Omega \to R^3$; hence $p(\Omega)$ is a smooth surface in R^3 for every $p \in \mathscr{D}$. Moreover, let $\varkappa$, $\varkappa \in \mathscr{D}$, be the known deformation. We shall assume that the material of fibres is elastic and that $\varkappa$ corresponds to the natural state of the fibre lattice under consideration. The Lagrangian (plane) strain tensor will be defined by $E(X) = 0.5[\nabla p^T(X)\nabla p(X) - \nabla\varkappa^T(X)\nabla\varkappa(X)]$, $X \in \Omega$; note that the material gradients $\nabla p(X)$, $\nabla\varkappa(X)$ are represented here by 3×2 matrices. Putting $G_A(X) \equiv t_A(X) \otimes t_A(X)$ we define m scalar fields $\varepsilon_A(X) \equiv \operatorname{tr}[G_A(X)E(X)]$, $A = 1, \ldots, m$, $X \in \Omega$. Hence the strains $e_A(X)$ in the fibres of the A-th family are given by

$$e_A(X) = \begin{cases} \varepsilon_A(X) & \text{if} \quad \varepsilon_A(X) \geqslant 0, \\ 0 & \text{if} \quad \varepsilon_A(X) < 0. \end{cases} \tag{2.1}$$

The formula (2.1) describes the fact that it is not possible to realize any shortening of a fibre which is unable to sustain compressive forces.

Let $\sigma_A = \sigma_A(X)$ stand for a tension at point X and in the A-th family of fibres. The condition $\sigma_A \geqslant 0$ holds for $A = 1, \ldots, m$ and every $X \in \Omega$. We shall assume that for $\sigma_A > 0$ the material of the fibre can be treated as elastic with the strain-stress relation given by $\varepsilon_A = \varphi_A(X, \sigma_A)$. Putting $K_A(X, \sigma) \equiv \partial\varphi\,(X, \sigma)/\partial\sigma$, $\sigma \geqslant 0$, and defining $\dot\sigma_A$, $\dot\varepsilon_A$ as the values of the stress and strain rate, respectively, we shall assume that for every $(\varepsilon_A, \dot\varepsilon_A, \sigma_A, \dot\sigma_A) \in R^4$ the following condition

$$\begin{aligned} (2.2) \quad & \big[[\varepsilon_A > 0] \wedge [\sigma_A > 0] \wedge [\dot\sigma_A \in R] \wedge [\dot\varepsilon_A = K_A(X, \sigma_A)\dot\sigma_A]\big] \\ & \vee \big[[\varepsilon_A = 0] \wedge [\sigma_A = 0] \wedge [\dot\sigma_A > 0] \wedge [\dot\varepsilon_A = K_A(X, 0)\sigma_A]\big] \\ & \vee \big[[\varepsilon_A = 0] \wedge [\sigma_A = 0] \wedge [\dot\sigma_A = 0] \wedge [\dot\varepsilon_A \leqslant 0]\big] \\ & \vee \big[[\varepsilon_A < 0] \wedge [\sigma_A = 0] \wedge [\dot\sigma_A = 0] \wedge [\dot\varepsilon_A \in R]\big], \qquad A = 1, \ldots, m, \end{aligned}$$

holds; at the same time the formula (2.1) has to be taken into account. It can be shown that the condition (2.2) is equivalent to ([1]):

$$(2.3)\quad \dot{\varepsilon}_A - K_A(X, \sigma_A)\dot{\sigma}_A \in N_{T_{\Sigma(\varepsilon_A)}(\sigma_A)}(\dot{\sigma}_A), \qquad \Sigma(\varepsilon) = \begin{cases} \overline{R}_+ & \text{if} \quad \varepsilon \geqslant 0, \\ \{0\} & \text{if} \quad \varepsilon < 0, \end{cases} \quad A = L, \dots, m.$$

The relation (2.3) can also be transformed to the form

$$(2.4)\quad \begin{aligned} &\dot{\varepsilon}_A = K_A(X, \sigma_A)\dot{\sigma}_A + \lambda_A, \quad \lambda_A(\tau - \sigma_A) \leqslant 0, \quad \forall \tau \in \Sigma(\varepsilon_A), \\ &\dot{\sigma}_A \lambda_A = 0, \quad \dot{\sigma}_A \in T_{\Sigma(\varepsilon_A)}(\sigma_A), \quad A = 1, \dots, m. \end{aligned}$$

In the foregoing formulas the subsets $\Sigma(\varepsilon_A)$ of R have to be interpreted as the sets of all σ_A which are admissible for the prescribed value of ε_A.

The constitutive relations (2.3) and (2.4) can be easily modified by assuming that there exist plastic deformations $\varepsilon_A^p(X)$ and putting

$$(2.5)\qquad \Sigma(\varepsilon_A) = \begin{cases} [0, \ \sigma_A^p(X)] & \text{if} \quad \varepsilon_A \geqslant \varepsilon_A^p(X), \\ \{0\} & \text{if} \quad \varepsilon_A < \varepsilon_A^p(X), \end{cases} \quad A = 1, \dots, m,$$

where $\sigma_A^p(X)$ is the known yield stress. Taking into account Eq. (2.5), we arrive at the case of what can be called elastic-plastic-slender deformations which are described by the formulas (2.4) and (2.5). In what follows we shall restrict ourselves to the case of elastic-slender deformations which are described by the formulas (2.4) in which $\Sigma(\varepsilon)$ is defined by Eq. $(2.3)_2$.

It must be emphasized that the "slender" models of fibres proposed in the paper cannot be used is such problems as the bending or buckling of fabrics in which the compressive forces also have to be taken into account.

3. Incremental theory of finite elastic-slender deformations

Let the mapping $t \to p(\cdot, t) \in \mathscr{D}$ describes a certain motion of the material surface under consideration and let $\sigma_A(\cdot, t)$, $A = 1, \dots, m$, be the stress distribution at the time instant t, defined almost everywhere on Ω. Moreover, let $b(X, t) \in R^3$ be the external body and inertia force density (related to Ω and defined a.e. on Ω) and let $s(X, t) \in R^2$ be the boundary traction density (related to $\partial\Omega$ and defined a.e. on $\partial\Omega$). Let us also introduce a linear topological space V of the (sufficiently regular) functions $v\colon \Omega \to R^3$, which will be treated below as the test functions. The equations of motion of the material surface under consideration will be postulated in the form ([2])

$$(3.1)\quad \int_\Omega \sum_{A=1}^{m} \sigma_A(X, t) G_A^{\alpha\beta}(X) p^k_{,\alpha}(X, t) v_{k,\beta}(X)\, dv(X)$$
$$= \oint_{\partial\Omega} p^k_{,\alpha}(X, t) s^\alpha(X, t) v_k(X)\, da(X) + \int_\Omega [p^k_{,\alpha}(X, t) b^\alpha(X, t) + n^k(X, t) b^3(X, t)] v_k(X)\, dv(X),$$

([1]) For any closed, convex and nonempty subset Δ of R^n, we define $N_\Delta(x) := \{z \in R^n | z \cdot (w - x) \leqslant 0, \forall w \in \Delta\}$ for $x \in \Delta$; $N_\Delta(x) = \phi$ for $x \in R^n \setminus \Delta$, $T_\Delta(x) := \{z \in R^n | \ z \cdot w \leqslant 0 \ \forall w \in N_\Delta(x)\}$ for $x \in \Delta$; $T_\Delta(x) = \phi$ for $x \in R^n \setminus \Delta$ and refer $N_\Delta(x)$, $T_\Delta(x)$ to as a normal and tangent cone, respectively, to a set Δ at a point $x \in R^n$.

([2]) The indices α β, run over the sequence 1, 2, while the index k runs over 1, 2, 3; summation convention holds. The symbol $n(X, t)$, $X \in \Omega$, stands for the unit vector normal to the surface $p(\Omega, t)$ in R^3 at point $p(X, t)$.

which has to hold for every $v \in V$ and every time instant t, for which all integrals exist. The equations of motion (3.1) can be expressed in a simple form. To this end we introduce the linear topological space W of the functions $w\colon \Omega \to R^m$ and the linear continuous operators (defined for every $p \in \mathscr{D}$) $\Lambda_p\colon V \to W$ given by

$$(3.2)\qquad (\Lambda_p v)(X) \equiv \big(\operatorname{tr}[\nabla p(X) G_1(X) \nabla v^T(X)], \ldots, \operatorname{tr}[\nabla p(X) G_m(X) \nabla v(X)]\big).$$

We shall also introduce dual pairings $(V, \langle\ ,\ \rangle, V^*)$, $(W, (\ ,\ ), W^*)$ and the operator $\Lambda_p^*\colon W^* \to V^*$, conjugate to Λ_p, i.e. such that $(\Lambda_p v, S) = \langle v, \Lambda_p^* S\rangle$ hold for every $(v, S) \in V \times W^*$. Defining the functionals

$$(3.3)\qquad \langle v, f_p(t)\rangle \equiv \oint_{\partial\Omega} p^k_{\cdot\alpha}(X, t) s^\alpha(X, t) v_k(X)\, da(X)$$

$$+ \int_\Omega [p^k_{\cdot\alpha}(X, t) b^\alpha(X, t) + n^k(X, t) b^3(X, t)] v_k(X)\, dv(X),$$

$$\big(w, S(t)\big) \equiv \int_\Omega \sum_{A=1}^{m} \omega_A(X) \sigma_A(X, t)\, dv(X), \qquad v \in V, \qquad w \in W,$$

we shall rewrite Eq. (3.1) in the form $(\Lambda_{p(t)} v, S(t)) = \langle v, f_p(t)\rangle$, $\forall v \in V$, $p(t) \equiv p(\cdot, t)$, or

$$(3.4)\qquad \Lambda^*_{p(t)} S(t) = f_p(t),$$

which has to hold for every time instant t for which the functionals $S(t)$, $f_p(t)$ exist.

The equations of motion (3.1) or (3.4) and constitutive relations (2.3) or (2.4) in which

$$\varepsilon_A = \varepsilon_A(X, t) = \frac{1}{2} \operatorname{tr}\big[G_A(X) \big(\nabla p^T(X, t) \nabla p(X, t) - \nabla \varkappa^T(X) \nabla \varkappa(X)\big)\big],$$

$$(3.5)\qquad \dot\varepsilon_A = \dot\varepsilon(X, t) = \operatorname{tr}\big[G_A(X) \big(\nabla p^T(X, t) \nabla \dot p(X, t)\big)\big],$$

$$\sigma_A = \sigma_A(X, t), \qquad \dot\sigma_A = \dot\sigma_A(X, t), \qquad A = 1, \ldots, m,$$

constitute the governing relations of the theory of finite elastic-slender deformations.

4. Incremental theory od small elastic-slender deformations

Theories of small deformations of fibrous slender materials are based on the following assumptions:

1. The fields $\varepsilon_A(X, t)$, $\dot\varepsilon_A(X, t)$ in constitutive relations can be approximated by the fields obtained from the RHS of Eqs. $(3.5)_{1,2}$ by linearization with respect to the displacement gradients $\nabla[p(X, t) - \varkappa(X)]$,

2. The equations of motion (3.1) and (3.4) can be approximated by the equations obtained from Eq. (3.1) by linearization with respect to

$$\nabla[p(X, t) - \varkappa(X)],\ \sigma_A(X, t),\ s^\alpha(X, t),\ b^\alpha(X, t),\ b^3(X, t),$$

3. The displacement field $p(\cdot, t) - \varkappa(\cdot)$ for every time instant t, can be approximated by a certain field $u(\cdot, t)$ from the linear topological space V.

From Assumption 2 it follows that Eq. (3.4) has to be replaced by

$$(4.1)\qquad \Lambda^*_\varkappa S(t) = f_\varkappa(t),$$

where $\Lambda_\varkappa: V \to W$ is defined by Eq. (3.2) for $p = \varkappa$. From Assumptions 1, 3 we obtain that

$$(4.2)\qquad \begin{aligned} \varepsilon_A &= \varepsilon_A(X, t) \equiv \operatorname{tr}[G_A(X) L_\varkappa u(X, t)], \\ \dot{\varepsilon}_A &= \dot{\varepsilon}_A(X, t) \equiv \operatorname{tr}[G_A(X) L_\varkappa \dot{u}(X, t)], \qquad A = 1, \ldots, m, \end{aligned}$$

where

$$(4.3)\qquad L_\varkappa u \equiv \frac{1}{2}(\nabla\varkappa^T \nabla u + \nabla u^T \nabla \varkappa).$$

Here $L_\varkappa$ can be treated as the linear continuous operator defined on V and with values in a certain linear topological spaces Y of the functions $D: \Omega \to R^{(2\times 2)}$. It must be emphasized that in the theories of small deformations the fields $\varepsilon_A(\cdot, t)$, $\dot{\varepsilon}_A(\cdot, t)$ are defined by the formulas (4.2), i.e. they are not the strain and the strain rate fields, respectively. Let us also observe that the functional $f_\varkappa(t)$ in Eq. (4.1), defined by (cf. the formula $(3.3)_1$ (³))

$$\langle v, f_\varkappa(t)\rangle \equiv \oint_{\partial\Omega} \varkappa^k_{,\alpha}(X) s^\alpha(X, t) v_k(X)\, da(X) + \int_\Omega [\varkappa^k_{,\alpha}(X) b^\alpha(X, t) + N^k(X) b^3(X, t)] v_k(X)\, dv(X),$$

is independent of $u(\cdot, t)$.

The equation of motion (4.1) and constitutive relations (2.3) or (2.4) in which $\sigma_A = \sigma_A(X, t)$, $\dot{\sigma}_A \equiv \dot{\sigma}_A(X, t)$ and ε_A, $\dot{\varepsilon}_A$ are defined now by the formulas (4.2), constitute the governing relations of the incremental theory of small elastic-slender deformations.

5. Theory of linear-elastic slender deformations

We shall postulate now that the assumptions 1, 2, 3 of Section 4 hold and that for every $\sigma_A > 0$, strain-stress relations have the linear form $\varepsilon_A = K_A(X)\sigma_A$, $K_A(X)$, $A = 1, \ldots, m$, being the known positive numbers. Then, due to the slenderness of fibres, for every $(\varepsilon_A, \sigma_A, \dot{\sigma}_A) \in R^3$ the following condition holds:

$$(5.1)\qquad \begin{aligned} &[[\sigma_A > 0] \wedge [\dot{\sigma}_A \in R] \wedge [\varepsilon_A = K_A(X)\sigma_A]] \vee [[\sigma_A = 0] \wedge [\dot{\sigma}_A > 0] \wedge [\varepsilon_A = 0]] \\ &\vee [[\sigma_A = 0] \wedge [\dot{\sigma}_A = 0] \wedge [\varepsilon_A \leqslant 0]], \qquad A = 1, \ldots, m. \end{aligned}$$

The foregoing condition is equivalent to

$$(5.2)\qquad \varepsilon_A - K_A(X)\sigma_A \in N_{T_{\bar{R}_+}(\sigma_A)}(\dot{\sigma}_A), \qquad A = 1, \ldots, m,$$

or to

$$\varepsilon_A = K_A(X)\sigma_A + \lambda_A, \quad \lambda_A \in N_{\bar{R}_+}(\sigma_A), \quad \dot{\sigma}_A \lambda_A = 0, \quad \dot{\sigma}_A \in T_{\bar{R}_+}(\sigma_A), \quad A = 1, \ldots, m.$$

The equation of motion (4.1) and constitutive relations (5.2) in which $\sigma_A = \sigma_A(X, t)$, $\dot{\sigma}_A = \dot{\sigma}_A(X, t)$ and ε_A is defined by Eq. $(4.2)_1$, are the governing relations of the theory of linear-elastic slender deformations.

(³) Here $N(X)$ stands for a unit vector normal to the surface $\varkappa(\Omega)$ in R^3 at point $\varkappa(X)$, $X \in \Omega$.

6. Modelling of rigid-slender deformations

Now assume that the fibres can be treated as inextensible, i.e. they are subject to the condition $\varepsilon_A \leqslant 0$. Taking into account the slenderness of fibres, for every $(\varepsilon_A, \sigma_A, \dot{\varepsilon}_A, \dot{\sigma}_A) \in R^4$ we obtain

$$(6.1)\quad [[\varepsilon_A = 0]\wedge[\sigma_A > 0]\wedge[\dot{\sigma}_A \in R]\wedge[\dot{\varepsilon}_A = 0]] \vee [[\varepsilon_A = 0]\wedge[\sigma_A = 0]\wedge[\dot{\sigma}_A > 0]\wedge[\dot{\varepsilon}_A = 0]] \vee [[\varepsilon_A = 0]\wedge[\sigma_A = 0]\wedge[\dot{\sigma}_A = 0]\wedge[\dot{\varepsilon}_A \leqslant 0]] \vee [[\varepsilon_A < 0]\wedge[\sigma_A = 0]\wedge[\dot{\sigma}_A = 0]\wedge[\dot{\varepsilon}_A \in R]],\quad A = 1, \ldots, m.$$

Hence

$$(6.2)\quad \dot{\varepsilon}_A \in N_{T_{\Sigma(\varepsilon_A)}(\sigma_A)}(\dot{\sigma}_A),\quad \Sigma(\varepsilon) \equiv \begin{cases} \bar{R}_+ & \text{if}\quad \varepsilon = 0,\\ \{0\} & \text{if}\quad \varepsilon < 0,\end{cases}$$

or

$$(6.3)\quad \dot{\varepsilon}_A(\tau - \sigma_A) \leqslant 0\quad \forall \tau \in \Sigma(\varepsilon_A),\quad \dot{\sigma}_A \dot{\varepsilon}_A = 0,\quad \dot{\sigma}_A \in T_{\Sigma(\varepsilon_A)}(\sigma_A),\quad A = 1, \ldots, m.$$

are the constitutive relations of rigid-slender deformations. It can be easily observed that the relations (6.2) and (6.3) constitute a special case of Eqs. (2.3) and (2.4) in which $K_A \equiv 0$. For small deformations we obtain the conditions (5.1) and (5.2) in which $K_A \equiv 0$. Thus the constitutive relations for small rigid-slender deformations will be given by

$$(6.4)\quad \varepsilon_A \in N_{\bar{R}+}(\sigma_A),\quad \dot{\sigma}_A \varepsilon_A = 0,\quad \dot{\sigma}_A \in T_{\bar{R}+}(\sigma_A),\quad A = 1, \ldots, m.$$

Note that the condition $\varepsilon_A \in N_{\bar{R}+}(\sigma_A)$ in the relations (6.4) is equivalent to the condition $\sigma_A \in N_{\bar{R}-}(\varepsilon_A)$.

Using the approach analogous to that of Sects. 3, 4, 5, but taking into account the constitutive relations (6.3) or (6.4) (instead of the constitutive relations (2.4) or (5.2), respectively) we obtain the governing relations of what can be called:

1. The theory of finite rigid-slender deformations, based on the equation of motion (3.1) or (3.4), constitutive relations (6.3) and formulas (3.5).

2. The incremental theory of small rigid-slender deformations, based on the equation of motion (4.1), constitutive relations (6.3) and formulas (4.2).

3. The theory of small rigid-slender deformations, based on the equation of motion (4.1), constitutive relations (6.4) and formula $(4.2)_1$.

It must be emphasized that the formulation of special problems within the theories formulated in Sects. 3, 4, 5 as well as the theories mentioned above requires also informations about the interrelation between the material body under consideration and its exterior. The general form of this interrelation will be proposed in the subsequent section.

7. External constraints and loadings

Let Γ_a, $a = 1, \ldots, n$, be the known parts of $\bar{\Omega}$ and $g_a\colon \Gamma_a \to R^2$, $a = 1, \ldots, n$, be the known fields of the unit vectors. Taking into account the physical premises concerning fibrous materials, we shall postulate the external kinematic constraints in the form

$$g_a^{\alpha}(X) p^k_{,\alpha}(X, t) \dot{p}_k(X, t) \geqslant 0,\quad X \in \Gamma_a,\quad a - 1, \ldots, n.$$

Putting

(7.1) $\Xi(p(t)) := \{v \in V | v_k(X) p^k_{,\alpha}(X,t) g^\alpha_a(X) \geq 0$ for a.e. $X \in \Gamma_a,\ a = 1, \ldots, n\}$

and using the general theory of constraints, [6], we shall assume that the functional $f_p(t)$, $f_p(t) \in V^*$ determining the external forces in the equation of motion (3.4) is given by ([4])

(7.2) $$f_p(t) = l_p(t) + r(t), \quad r(t) \in -\partial\chi_{\Xi(p(t))}(\dot{p}(t)),$$

where $l_p(t)$ are the loadings and $r(t)$ are reactions to the kinematic constraint $\dot{p}(t) \in \Xi(p(t))$. The formula (7.2) will be applied in theories of finite deformations.

On passing to theories of small deformations, we have to replace the deformation gradients $p^k_{,\alpha}(X,t)$ in Eq. (7.1) by the gradients $\varkappa^k_{,\alpha}(X)$; such an approximation is consistent with the assumptions mentioned at the beginning of Sect. 4. Putting

(7.3) $$\Xi := \{v \in V | v_k(X) \varkappa^k_{,\alpha}(X) g^\alpha_a(X) \geq 0 \quad X \in \Gamma_a,\ a = 1, \ldots, n\}$$

we shall assume that the functional $f_\varkappa(t), f_\varkappa(t) \in V^*$ in the equation of motion (4.1) is given by

(7.4) $$f_\varkappa(t) = l_\varkappa(t) + r(t), \quad r(t) \in -\partial\chi_\Xi(\dot{u}(t)), \quad u(t) \equiv u(\cdot, t),$$

where $l_\varkappa(t)$ are the known loadings. The formula (7.4) will be applied in incremental theories of small deformations. In the nonincremental theories we shall postulate the external constraints in the form $u(t) \in \Xi$. Hence

(7.5) $$f_\varkappa(t) = l_\varkappa(t) + r(t), \quad r(t) \in -\partial\chi_\Xi(u(t)),$$

has to be substituted into the equation of motion (4.1) if we deal with the theory of small elastic-slender or rigid-slender deformations. Postulating Eqs. (7.2), (7.4) or (7.5), we have tacitly assumed that the problems under consideration are quasi-static problems, i.e. we have neglected inertia forces. It must be also emphasized that the functionals $r(t)$, $r(t) \in V^*$ can have a more general representation than that given by the formula $(3.3)_1$; that is why the equation of motion has to be postulated in the form given by Eqs. (3.4) or (4.1) instead of that given by Eq. (3.1). The functionals $l_p(t)$ have representations of the form $(3.3)_1$ and hence they depend on the deformation $p(t)$; the functionals $l_\varkappa(t)$ have representations of the form (4.4) in which $\varkappa^k_{,\alpha}\alpha(\cdot)$ and $N^k(\cdot)$ are known.

8. On the formulation of boundary-value problems

Using the results of Sects. 2–7 we can now formulate the basic boundary-value problems for fibrous slender materials. We confine ourselves to the quasi-static problems only and assume that the initial values of the functions $t \to (p(t), \sigma(t))$ or $t \to (u(t), \sigma(t))$, where $\sigma(t) \equiv (\sigma_1(\cdot, t), \ldots, \sigma_m(\cdot, t))$ are known. We are not going to investigate the formulated problems from the point of view of the existence and uniqueness theorems; hence the spaces V, V^* and W, W^* are not made precise here (the existence theorems concerning plane static problems will be studied in [7]).

([4]) The symbol χ_Ξ stands for the indicator function of the set Ξ in V and $\partial\chi_\Xi(v)$ is the subdifferential of χ_Ξ at $v \in V$; here Ξ is a nonempty closed, and convex cone in V. The subset $\partial\chi_\Xi(v)$ of V^* represents the cone normal to Ξ at $v \in V$; it is the set of all reactions due to the constraints $v \in \Xi$, while $-\partial\chi_\Xi(v)$ the set of all reactions maintaining the constraints (external reactions).

1. Incremental theory of finite elastic-slender deformations. We look for the function $t \to (p(t), \sigma(t))$ which satisfies the following relations:

$$\begin{gathered}\Lambda^*_{p(t)} S(t) - l_p(t) \in -\partial\chi_{\Xi(p(t))}(\dot{p}(t)),\\ \dot{\varepsilon}_A(X,t) - K_A(X, \sigma_A(X,t))\dot{\sigma}_A(X,t) \in N_{T_{\Sigma(\varepsilon_A(X,t))}(\sigma_A(X,t))}(\dot{\sigma}_A(X,t));\\ A = 1, \dots, m; \quad X \in \Omega,\end{gathered} \tag{8.1}$$

where

$$\varepsilon_A(X,t) \equiv \frac{1}{2}\operatorname{tr}\{G_A(X)[\nabla p^T(X,t)\nabla p(X,t) - \nabla\varkappa^T(X)\nabla\varkappa(X)]\},$$

and where $S(t)$ is given by Eq. $(3.3)_2$. The functionals $l_p(t)$ are assumed to be uniquely determined by the deformations $p(t)$; the mappings $\Xi(\cdot)$, $K_A(\cdot)$, $A = 1, \dots, m$ are assumed to be known.

2. Incremental theory of small elastic-slender deformations. We look for the function $t \to (u(t), \sigma(t))$ which satisfies the relations

$$\begin{gathered}\Lambda^*_\varkappa S(t) - l_\varkappa(t) \in -\partial\chi_\Xi(u(t)),\\ \dot{\varepsilon}_A(X,t) - K_A(X, \sigma_A(X,t))\dot{\sigma}_A(X,t) \in N_{T_{\Sigma(\varepsilon_A(X,t))}(\sigma_A(X,t))}(\sigma_A(X,t));\\ A = 1, \dots, m; \quad X \in \Omega,\end{gathered} \tag{8.2}$$

where

$$\varepsilon_A(X,t) \equiv \frac{1}{2}\operatorname{tr}\{G_A(X)[\nabla\varkappa^T(X)\nabla u(X,t) + \nabla u^T(X,t)\nabla\varkappa(X)]\},$$

and where $S(t)$ is given by Eq. $(3.3)_2$. The functionals $l_\varkappa(t)$ and mappings $K_A(\cdot)$, $A = 1, \dots, m$, as well as the subset Ξ of V are assumed to be known.

3° Theory of small elastic-slender deformations. We look for the function $t \to (u(t), \sigma(t))$ which satisfies the relations

$$\begin{gathered}\Lambda^*_\varkappa S(t) - l_\varkappa(t) \in -\partial\chi_\Xi(u(t)),\\ \varepsilon_A(X,t) - K_A(X)\sigma_A(X,t) \in N_{T_{\bar{R}_+}(\sigma_A(X,t))}(\dot{\sigma}_A(X,t)); \quad A = 1, \dots, m; \quad X \in \Omega,\end{gathered} \tag{8.3}$$

where

$$\varepsilon_A(X,t) \equiv \frac{1}{2}\operatorname{tr}\{G_A(X)[\nabla\varkappa^T(X)\nabla u(X,t) + \nabla u^T(X,t)\nabla\varkappa(X)]\},$$

and where $S(t)$ is given by the relations $(3.3)_2$. The functionals $l_\varkappa(t)$, functions $K_A(\cdot)$, $A = 1, \dots, m$, and subset Ξ of V are assumed to be known.

Substituting $K_A \equiv 0$, $A = 1, \dots, m$, into the relations (8.1)–(8.3), we arrive at the corresponding *theories of the rigid-slender deformations*. Taking into account the equivalent forms of the constitutive relations in the relations (8.1)–(8.3) (cf. the formulas (2.4) and (5.2)), we conclude that the problems under consideration are governed by systems of quasi-variational inequalities (in the incremental problems) or variational inequalities (in the nonincremental problems, i.e. problems governed by relations of the form (8.3)).

Now assume that $\mathscr{D}$ is a set of plane deformations $p: \Omega \to R^2$, cf. Sect. 2; hence every $\nabla p(X)$, $X \in \Omega$ is a 2×2 nonsingular matrix and V can be assumed as a linear topological space of (sufficiently regular) functions $V: \Omega \to R^2$. The formulas (3.1), (3.3), (4.4), (7.1) and (7.3) hold under the assumption that the index k runs over 1, 2 and that $b^3 \equiv 0$. Thus we obtain the class of what will be called the plane problems.

As examples of the plane problems we shall formulate the static plane problems of the theories of small elastic-slender and rigid-slender deformations. We are going to show that the forementioned problems are described by systems of two variational inequalities. Let $\varkappa = id$ and define $\Lambda = \Lambda_\varkappa$ for $\varkappa = id$. From the relations (8.3) we obtain (cf. also the formulas (5.2))

$$\text{(8.4)} \qquad \Lambda^* S - l \in -\partial\chi_\Xi(u),$$
$$\varepsilon_A(X) - K_A(X)\sigma_A(X) \in N_{\bar{R}_+}(\sigma_A(X)), \qquad A = 1, \ldots, m, \qquad X \in \Omega,$$

where

$$\varepsilon_A(X) \equiv \mathrm{tr}[G_A(X)Lu(X)], \qquad Lu(X) \equiv \frac{1}{2}[\nabla u(X) + \nabla u^T(X)],$$
$$\Lambda u(X) = \big(\mathrm{tr}[G_1(X)Lu(X)], \ldots, \mathrm{tr}[G_m(X)Lu(X)]\big).$$

Let Δ be the closed convex cone in W^* defined by

$$\text{(8.5)} \qquad \Delta := \{S \in W^* | \sigma_A(X) \geqslant 0 \quad \text{for a.e.} \quad X \in \Omega, \ A = 1, \ldots, m\},$$

where

$$(w, S) \equiv \int_\Omega \sum_{A=1}^{m} w_A(X)\sigma_A(X)dv(X), \qquad \forall w \equiv \big(w_1(\cdot), \ldots, w_m(\cdot)\big) \in W.$$

Let K: $W^* \to W$ be the linear monotone operator such that

$$\text{(8.6)} \qquad (KS, \tilde{S}) = \int_\Omega \sum_{A=1}^{m} K_A(X)\sigma_A(X)\tilde{\sigma}_A(X)dv(X), \qquad S, \tilde{S} \in W^*.$$

Using Eqs. (8.5) and (8.6), the constitutive relations $(8.4)_2$ can be written down in the form

$$\Lambda u - KS \in \partial\chi_\Delta(S),$$

where χ_Δ is the indicator function of the closed convex cone Δ in W^*, and $\partial\chi_\Delta(S)$ is its subdifferential at $S \in W^*$. Hence we conclude that the static plane problem of the theory of small elastic-slender deformations can be stated as follows: find $(u, S) \in V \times W^*$ such that

$$\text{(8.7)} \qquad \begin{aligned} &\Lambda^* S - l \in -\partial\chi_\Xi(u), \\ &\Lambda u - KS \in \partial\chi_\Delta(S), \end{aligned}$$

holds. We have tacitly assumed that in every problem under consideration the objects $l \in V^*$, K: $W^* \to W$, $\Xi \subset V$ and $\Delta \subset W^*$ are known. Introducing the linear topological space Y of the functions D: $\Omega \to R^{(2\times 2)}$ (cf. Sect. 4) such that $LV \subset Y$ and the linear continuous operator G: $Y \to W$, defined by

$$GD(X) \equiv \big(\mathrm{tr}[G_1(X)D(X)], \ldots, \mathrm{tr}[G_m(X)D(X)]\big),$$

we obtain $\Lambda = GL$. Hence the relations (8.7) can also be written down in the alternative form

$$(8.8)\qquad \begin{aligned} L^*G^*S-l &\in -\partial\chi_{\Xi}(u),\\ GLu-KS &\in \partial\chi_{\Delta}(S), \end{aligned}$$

in which the geometry of the fibre structure of a material is described by mapping $G: Y \to W$.

It can be shown that for the rigid-slender static deformations we obtain the following problem: find $(u, S) \in V \times W^*$ such that

$$(8.9)\qquad \begin{aligned} \Lambda^*S-l &\in -\partial\chi_{\Xi}(u),\\ \Lambda u &\in \partial\chi_{\Delta}(S), \end{aligned}$$

holds. This is a special case of the relations (8.7) in which $K \equiv 0$. But $w \in \partial\chi_{\Delta}(S)$ implies $S \in \partial\chi_{\Delta}^*(w) = \partial\chi_{\Delta*}(w)$ where Δ^* is a cone conjugate to Δ, and given by

$$\Delta^* := \{w \in W | w = \big(w_1(\cdot), \ldots, w_m(\cdot)\big),\ w_A(X) \leqslant 0 \quad \text{for a.e.} \quad X \in \Omega,\ A = 1, \ldots, m\}.$$

Hence the problems of plane rigid-slender static deformations are governed by the system of two following variational inequalities:

$$(8.10)\qquad \begin{aligned} \Lambda^*S-l &\in -\partial\chi_{\Xi}(u),\\ S &\in \partial\chi_{\Delta^*}(\Lambda u). \end{aligned}$$

The detailed analysis of the static problems formulated here will be given in [7].

9. Examples of solutions

To illustrate the general considerations developed in the paper, we shall analyse the plane, axially-symmetric problem for a material made of two families of slender fibres and subject to the radial boundary tractions p_a, p_b (cf. Fig. 2). We introduce the polar

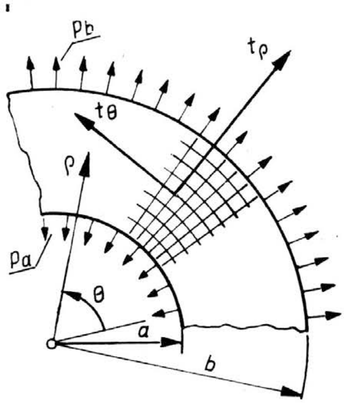

FIG. 2.

reference frame ϱ, θ on the plane OX^1X^2, putting $X^1 = \varrho\cos\theta$, $X^2 = \varrho\sin\theta$ for every $X \in (X^1, X^2) \in \Omega$, Ω being the circular annulus: $a < \sigma < b, 0 \leqslant \theta \leqslant 2\pi$. The basic unknowns are: radial displacements $u = u(\varrho)$ and tensions $\sigma_\varrho = \sigma_\varrho(\varrho)$, $\sigma_\theta = \sigma_\theta(\varrho)$ in the radial and circumferential fibres, respectively. The problem of finding $u(\cdot)$, $\sigma_\varrho(\cdot)$, $\sigma_\theta(\cdot)$ will be formulated as a problem of small elastic-slender deformations, which is governed by (8.4). We are not going to introduce any constraint for deformations; hence the relation $(8.4)_1$

reduces to $\Lambda^*S = l$. Under the known regularity conditions, the equilibrium equations $\Lambda^*S = l$ yield the well-known formulas

$$(\varrho\sigma_\varrho)_{,\varrho} - \sigma_\theta = 0 \quad \text{for} \quad \varrho \in (a, b), \quad \sigma_\varrho(a) = p_a, \quad \sigma_\varrho(b) = p_b. \tag{9.1}$$

We have assumed here that the subscript A runs over ϱ, θ. Putting $K \equiv K_\varrho = K_\theta$ and assuming that K is constant, we obtain from the relation $(8.4)_2$

$$\varepsilon_\varrho - K\sigma_\varrho \in N_{\overline{R}_+}(\sigma_\varrho), \qquad \varepsilon_\theta - K\sigma_\theta \in N_{\overline{R}_+}(\sigma_\theta). \tag{9.2}$$

Here and in what follows we neglect the argument ϱ in basic formulas. The fields $G_A(\cdot)$, $A = \varrho, \theta$, are now given by

$$G_\varrho = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad G_\theta = \begin{bmatrix} 0 & 0 \\ 0 & \varrho^2 \end{bmatrix}, \qquad \varrho \in (a, b).$$

Hence, from

$$\varepsilon_A \equiv \frac{1}{2}\operatorname{tr}[G_A(\nabla u + \nabla u^T)], \qquad A = \varrho, \theta,$$

we obtain the strain-displacement relations in the known form

$$\varepsilon_\varrho = u_{,\varrho} \qquad \varepsilon_\theta = \frac{u}{\varrho}. \tag{9.3}$$

Eliminating ε_ϱ, ε_θ from the relations (9.2) by means of the relations (9.3) we arrive at the system of equations and inequalities for the functions: $u(\varrho)$, $\sigma_\varrho(\varrho)$, $\sigma_\theta(\varrho)$, in which K, a, b, p_a, p_b, are assumed to be known. It can be easily shown that the solutions of this problem exist only if $p_a \geqslant 0$ and $p_b \geqslant 0$. Putting aside the detailed calculations, we shall confine ourselves to the final results.

1. Let us firstly assume that $p_a > 0$; then the existence and the form of the solutions depends on the ratio $p_b p_a^{-1}$. It can be shown that the following special cases have to be taken into account:

1.1. Case $p_b p_a^{-1} < ab^{-1}$: there are no solutions.

1.2. Case $p_b p_a^{-1} = ab^{-1}$: the solutions are given by

$$u = ap_a K \ln\frac{\varrho}{a} + D, \qquad \sigma_\varrho = p_a\frac{a}{\varrho}, \qquad \sigma_\theta = 0, \qquad \varrho \in (a, b),$$

where D is an arbitrary constant satisfying the condition

$$D \leqslant -ap_a K \ln\frac{b}{a}.$$

1.3. Case $ab^{-1} < p_b p_a^{-1} < 0.5\,[1 + (a/b)^2]$: there exists the "free boundary" $\varrho = r$, $r \in (a, b)$, determined by

$$\frac{r}{a} = \left(\frac{b}{a}\right)^2 \left[\frac{p_b}{p_a} - \sqrt{\left(\frac{p_b}{p_a}\right)^2 - \left(\frac{a}{b}\right)^2}\right],$$

such that for $\varrho \in (a, r)$ we have

$$u = ap_a K \ln\frac{\varrho}{r}, \qquad \sigma_\varrho = p_a\frac{a}{\varrho}, \qquad \sigma_\theta = 0,$$

and for $\varrho \in (r, b)$ we get

$$u = -\frac{AK}{\varrho} + CK\varrho, \quad \sigma_\varrho = \frac{A}{\varrho^2} + C, \quad \sigma_\theta = -\frac{A}{\varrho^2} + C,$$

where we have denoted

$$A \equiv \frac{r^2 b^2 (p_r - p_b)}{b^2 - r^2}, \quad C \equiv \frac{b^2 p_b - r^2 p_r}{b^2 - r^2}, \quad p_r = p_a \frac{a}{r}.$$

1.4. Case $p_b p_a^{-1} \geqslant 0.5[1 + (a/b)^2]$: the solution is given by

$$u = -\frac{AK}{\varrho} + CK\varrho, \quad \sigma_\varrho = \frac{A}{\varrho^2} + C, \quad \sigma_\theta = -\frac{A}{\varrho^2} + C, \quad \varrho \in (a, b),$$

where

$$A \equiv \frac{a^2 b^2 (p_a - p_b)}{b^2 - a^2}, \quad C \equiv \frac{b^2 p_b - a^2 p_a}{b^2 - a^2}.$$

2. Now assume that $p_a = 0$: then the existence and the form of solutions depends on the value of p_b. The following special cases have to be taken into account:

2.1. Case $p_b < 0$: there are no solutions.

2.2. Case $p_b = 0$: the solutions are given by

$$u = \varphi(\varrho), \quad \sigma_\varrho = 0, \quad \sigma_\theta = 0, \quad \varrho \in (a, b),$$

where $\varphi(\varrho)$ is an arbitrary function satisfying the conditions: $\varphi(\varrho) \leqslant 0$ and $\varphi_{,\varphi}(\varrho) \leqslant 0$ for every $\varrho \in (a, b)$.

2.3. Case $p_b > 0$: the solution is given by

$$u = -\frac{AK}{\varrho} + CK\varrho, \quad \sigma_\varrho = \frac{A}{\varrho^2} + C, \quad \sigma_\theta = -\frac{A}{\varrho^2} + C, \quad \varrho \in (a, b),$$

where

$$A \equiv \frac{a^2 b^2 p_b}{a^2 - b^2}, \quad C \equiv \frac{b^2 p_b}{b^2 - a^2}.$$

Now assume that the fibres are inextensible, passing to the theory of small rigid-slender deformations. The governing relations are (9.1), (9.3) and

$$\varepsilon_\varrho \in N_{\bar{R}_+}(\sigma_\varrho), \quad \varepsilon_\theta \in N_{\bar{R}_+}(\sigma_\theta), \tag{9.4}$$

or, equivalently,

$$\sigma_\varrho \in N_{\bar{R}_-}(\varepsilon_\varrho), \quad \sigma_\theta \in N_{\bar{R}_-}(\varepsilon_\theta). \tag{9.5}$$

It can be shown that the solutions exist under the condition that $p_b \geqslant p_a b^{-1} a$ and $p_a \geqslant 0$. For $p_b = p_a b^{-1} a$ and $p_a \geqslant 0$ we get

$$u = D, \quad \sigma_\varrho = p_a \frac{a}{\varrho}, \quad \sigma_\theta = 0, \quad \varrho \in (a, b),$$

where D is an arbitrary nonpositive constant. For $p_b > p_a b^{-1} a$ and $p_a \geqslant 0$, we obtain $u = 0$ and

$$\sigma_\varrho = \frac{a p_a}{\varrho} + \frac{1}{\varrho} \int_a^\varrho \sigma_\theta(\zeta) d\zeta, \quad \varrho \in (a, b),$$

where $\sigma_\theta(\cdot)$ is an arbitrary nonnegative function such that

$$bp_b - ap_a = \int_a^b \sigma_\theta \, d\varrho .$$

Hence we see that the solution of problems concerning rigid-slender deformations is not unique.

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DEPARTMENT OF MATHEMATICS, INFORMATICS AND MECHANICS
UNIVERSITY OF WARSAW.

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