Archives of Mechanics • Archiwum Mechaniki Stosowanej • 28, 3, pp. 333–352, Warszawa 1976

Kinetic theory of hydrodynamic flows

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We consider a dilute monatomic gas, flowing around a macroscopic heavy object. We derive an extended Boltzmann equation for the gas in the presence of the object, and show that the extended Boltzmann equation is equivalent to the usual Boltzmann equation, supplemented by the boundary conditions imposed on the distribution function at the surface of the object. We then proceed to solve the equation in two ways: (A) In the case when the mean free path of a gas molecule is small compared to the characteristic size of the object, we look for normal solutions of the Boltzmann equation where the distribution function is a functional of the local density, local velocity, and local temperature. This method of solution leads to the Navier-Stokes hydrodynamic equations, together with the boundary conditions that must be used to solve them. We also find that, beyond this order, boundary layer effects must be taken into account when formulating the boundary conditions satisfied by the hydrodynamic variables. We discuss the flow around a sphere, and consider the theory for stick flow, slip flow, and the region between pure stick and slip flows. (B) In the case when the mean free path is large compared to the size of the object, we solve the Boltzmann equation by an iteration method which is appropriate for describing the flow of a rarefied gas about the object, and we discuss the close relation between this iteration and the Knudsen iteration usually used in rarefied gas dynamics. We also show that the iteration method can also be used to treat the case when the mean free path is small compared to the object's size, and we derive Stokes' Law and the Lamb formula for the force on a sphere and a cylinder, respectively. By relating the terms in this iteration expansion to the dynamical events taking place in the fluid, we are able to discuss the dynamical origin of Stokes' Law and the Lamb formula.

Rozważany jest rozrzedzony gaz jednoatomowy opływający ważki obiekt makroskopowy. Wyprowadzono rozszerzone równanie Boltzmanna dla gazu wraz z obiektem i pokazano, że rozszerzone równanie Boltzmanna jest równoważne zwykłemu równaniu Boltzmanna, uzupełnionemu warunkami brzegowymi, nałożonymi na funkcję rozkładu na powierzchni obiektu. Z kolei przystąpiono do rozwiązania tego równania dwoma sposobami. (A) W przypadku, gdy droga swobodna cząsteczki gazu jest mała w porównaniu z charakterystycznym wymiarem obiektu, poszukuje się normalnych rozwiązań równania Boltzmanna, w których funkcja rozkładu jest funkcjonałem lokalnej gęstości, lokalnej prędkości i lokalnej temperatury. Ta metoda postępowania prowadzi do równań hydrodynamiki Naviera-Stokesa, które należy rozwiązywać wraz z warunkami brzegowymi. Wykazano również, że oprócz tego przy formułowaniu warunków brzegowych, spełnianych przez zmienne hydrodynamiczne, muszą być uwzględniane efekty warstwy przyściennej. Przedyskutowano przepływ wokół kuli i rozważono teorię dla przepływu z przyleganiem, przepływu poślizgowego oraz obszar między idealnym przepływem z przyleganiem i przepływem bez poślizgu. (B) W przypadku, gdy droga swobodna jest duża, równanie Boltzmanna rozwiązano metodą iteracyjną, która jest słuszna dla opisu przepływu rozrzedzonego gazu wokół obiektu oraz przedyskutowano ścisłą zależność między tą iteracją, a iteracją Knudsena, stosowaną zwykle w dynamice gazów rozrzedzonych. Pokazano również, że metoda iteracyjna może być wykorzystana w przypadku, gdy droga swobodna jest mała w porównaniu z rozmiarami ciała oraz wyprowadzono prawo Stokesa i wzór Lamba odpowiednio na siłę na kuli i na walcu. Przez powiązanie odpowiednich członów w tym rozwinięciu iteracyjnym z dynamicznymi efektami zachodzącymi w cieczy jesteśmy w stanie przedyskutować dynamiczne pochodzenie prawa Stokesa i wzoru Lamba.

Рассматривается разреженный, одноатомный газ, обтекающий весомый макроскопический объект. Выведено расширенное уравнение Больцмана для газа совместно с объектом и показано, что расширенное уравнение Больцмана эквивалентно обыкновенному уравнению Больцмана дополненному граничными условиями, наложенными на функцию распределения на поверхности объекта. В свою очередь приступается к решению этого уравнения двумя способами: (A) В случае, когда длина свободного пробега молекулы

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газа мала по сравнению с характеристическим размером объекта, ищутся нормальные решения уравнения Больцмана, в которых функция распределения является функционалом локальной плотности, локальной скорости и локальной температуры. Этот метод поступания ведет к уравнениям гидродинамики Навье-Стокса, которые следует решать совместно с граничными условиями. Тоже доказано, что кроме этого при формулировке граничных условий, удовлетворенных гидродинамическими переменными, должны быть учтены эффекты пограничного слоя. Обсуждено обтекание сферы и рассмотрена теория для обтекания с прилиганием, обтекания со скольжением, а также для области между идеальным обтеканием с прилиганием и обтеканием без скольжения. (Б) В случае, когда длина свободного пробега большая, уравнение Больцмана решено итерационным методом, который справедлив для описания обтекания объекта разреженным газом, а также обсуждена тесная зависимость между этой итерацией и итерацией Кнудсена, применяемой обычно в динамике разреженных газов. Показано тоже, что итерационный метод может быть использован в случае, когда длина свободного пробега мала по сравнению с размерами тела и выведен закон Стокса и формула Лэмба соответственно для силы на сфере и на цилиндре. Путем взаимосвязи соответствующих членов в этом итерационном разложении с динамическими эффектами, происходящими в жидкости, мы в состоянии обсудить динамическое происхождение закона Стокса и формулы Лэмба.

1. Introduction

CONSIDER the flow of a gas stream around a solid, heavy, macroscopic object. Let V be a characteristic velocity of the gas stream; c, the velocity of sound in the gas; R, a characteristic size of the object; l, the mean free path of a molecule in the gas; and a, the range of the intermolecular forces of the gas molecules. From these we construct the dimensionless quantities, the Mach number $\mathsf{M} = V/c$, the Knudsen number $\mathsf{K} = l/R$, and the ratio of the range of the intermolecular forces to the mean free path a/l. One of the central problems of the kinetic theory of gases is to determine the properties of the gas flow as a function of M, K, and a/l, assuming that the intermolecular force and the gas-solid interaction are known.

Here we will discuss one aspect of this problem, namely the kinetic theory for the Knudsen number dependence of the slow flow ($\mathsf{M} < 1$) of a dilute gas, for which $a/l < 1$, around a sphere or a cylinder. Even for these conditions there are a number of difficult problems, and we shall take the liberty of ignoring here the additional complications that arise when one attempts to extend the theory to higher densities or higher Mach numbers.

The starting point is, of course, the Boltzmann transport equation, satisfied by the one particle distribution function for the gas particles. To treat the flow around an object, the Boltzmann equation must be supplemented with boundary conditions satisfied by the distribution function at (i) the surface of the object (which takes into account the microscopic interaction mechanism between the gas particles and the object) and (ii) points very far from the object [1]. Then the quantities of interest for describing the flow are expressed in terms of the distribution function. Such quantities are $\varrho(\mathbf{r}, t)$, the average mass density of the gas at a point $\mathbf{r}$ at time t; $\mathbf{u}(\mathbf{r}, t)$, the average local velocity of the gas at $\mathbf{r}, t$; $\mathbf{F}(t)$, the force exerted on the object by the gas at t; and so on.

So far most of the work on these gas flows has been concentrated on the limiting cases where $\mathsf{K} \ll 1$, the Stokes or Clausius regime where the mean free path is much smaller than the characteristic size of the object, and $\mathsf{K} \gg 1$, the rarefied or Knudsen regime where the mean free path is much larger than the characteristic size of the object [2].

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For $K \ll 1$, the Boltzmann equation should lead to Stokes Law for the force on a sphere, say, and to the same results for $\varrho(\mathbf{r}, t)$ and $\mathbf{u}(\mathbf{r}, t)$ as given by the continuum Navier-Stokes hydrodynamic equations. In fact, one might be tempted to think that since the Chapman-Enskog normal solution [3] of the Boltzmann equation leads directly to the Navier-Stokes hydrodynamic equations, the derivation of Stokes Law should be simply an exercise in classical hydrodynamics. However, this procedure skips over some problems which have still not been solved in general. That is, the Chapman-Enskog procedure does not take into account the interactions of the fluid particles with the macroscopic object, but rather it ignores the possible presence of an object or boundaries, in general. As a result, it is not clear *a priori* that the Chapman-Enskog solution is a solution of the Boltzmann equation when the boundaries are properly taken into account. In fact, as we will discuss later, the Chapman-Enskog solution breaks down near the surface of the macroscopic object, and there are kinetic boundary layer effects which must be taken into account [4]. Furthermore, the Chapman-Enskog solution does not give the boundary conditions which the hydrodynamic variables must satisfy. Instead, these boundary conditions must be imposed on the variables without being related to the boundary conditions that the distribution function itself must satisfy.

It seems clear that a satisfactory derivation of Stokes Law along these lines must proceed by deriving both the Navier-Stokes hydrodynamic equations and the appropriate boundary conditions for the hydrodynamic variables from the Boltzmann equation and the boundary conditions satisfied by the distribution function. We shall discuss this problem in more detail in Sect. 3.

Another approach to the derivation of Stokes Law from the Boltzmann equation has been developed by CERCIGNANI and co-workers for simple model gases, by using variational methods [5]. In addition, SCHARF has obtained Stokes Law for the force on a sphere and Lamb's formula for the force per unit length on a cylinder by adding source terms to the Boltzmann equation [6]. However, he does not derive these source terms from the microscopic gas-solid interaction mechanism.

In any event, a discussion of the hydrodynamic results for the flow of a dilute gas around an object on the basis of the kinetic theory of gases (*i.e.*, from the Boltzmann equation) requires that one understand:

(a) The derivation of the Navier-Stokes hydrodynamic equations from the Boltzmann equation, together with the boundary conditions the hydrodynamic variables must satisfy.

(b) The role of higher order hydrodynamic equations, such as the Burnett and super-Burnett equations, in treating these gas flows (one must also have a derivation for the additional boundary conditions these higher order equations require).

(c) The dynamical origin of the hydrodynamical results, such as Stokes Law and the Lamb formula for the force on a cylinder, in terms of collision processes in the gas. In particular, one would like to understand the dramatic difference between the form of Stokes Law $\mathbf{F} = \zeta \mathbf{V}$ for the force on a sphere, which is linear in the flow velocity, and the form of Lamb's formula $\mathbf{F} = a\mathbf{V}/(b + \log \mathsf{M})$ for small M for the force per unit length on a cylinder whose long axis is perpendicular to the flow [7].

(d) How to extend the discussion of results for $K \ll 1$ to higher Knudsen numbers and eventually connect them to the results for rarefied gas flows.

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The other limit where the Boltzmann equation has been extensively applied to study gas flows is the case of rarefied flows where $K \gg 1$. Here one solves the Boltzmann equation by expanding the distribution function in terms of the inverse Knudsen number, explicitly taking into account collisions between gas molecules and between the molecules and the object. As an illustration, this expansion leads to an expression for the force F on a sphere, for $K \gg 1$, given by [8, 9a]

$$F/F_0 = 1 + a_1 K^{-1} + a_2' K^{-2} \log K^{-1} + a_2 K^{-2} + \dots , \tag{1.1}$$

where F_0 is the free molecular flow force, which is determined by single collisions between the gas molecules and the object, and F_0 is proportional to πR^2, the cross sectional area of the sphere, and to the flow velocity V. The coefficient a_1 is determined by sequences of three or more collisions taking place among two gas particles and the object, as illustrated in Fig. 1 (b,c,d.); a_2' is determined by sequences of four collisions among three particles and the sphere as illustrated in Fig. 1e; and a_2 is determined by collision sequences involving three or more particles and the sphere. For special models the coefficients a_1 and a_2 are known [10]. Moreover, the force F, for small V, appears to be linear in V

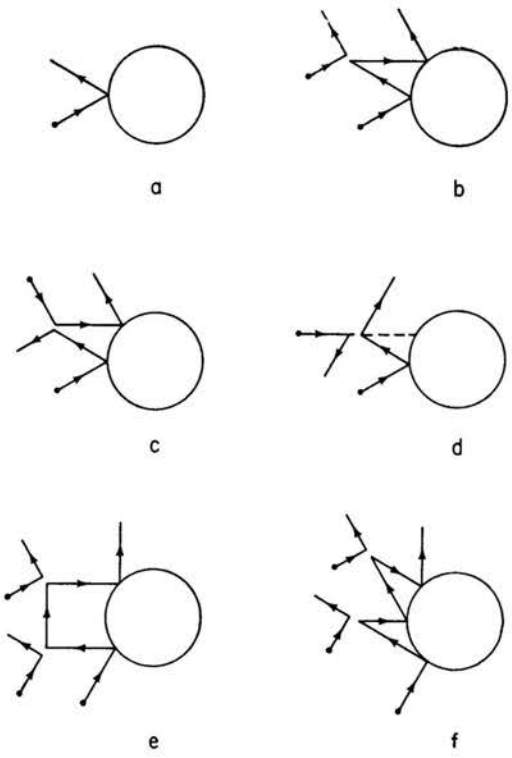

FIG. 1. The collisions of fluid particles with the sphere which are taken into account in the expansion of the force on the sphere in powers of the inverse Knudsen number. Fig. 1a represents the collision which is responsible for the free molecular flow force. Figs. 1b,c,d represent dynamical events which contribute to the K^{-1} correction of this value. Fig. 1d represents a process where the second fluid particle does not hit the sphere but would have done so, had the second collision not taken place. Fig. 1e represents events which contribute to order $K^{-2} \log K^{-1}$, and the events represented by Fig. 1f contribute to order K^{-2}.

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for all Knudsen numbers. There is a close relation between the form of the expansion of F as given by (1.1) and the density expansion of the transport coefficients. For example, the viscosity η of a moderately dense gas is given by

$$\eta/\eta_0 = 1+\eta_1(na^3)+\eta_2'(na^3)^2\log(na^3)+\eta_2(na^3)^2+ \dots , \tag{1.2}$$

where n is the number density of the gas and η_0 the viscosity at low density. This relation exists because the corresponding coefficients, (F_0, η_0), (a_1, η_1), (a_2', η_2'), etc., are determined by similar kinds of dynamical events [9, 10].

The force per unit length, f, on a cylinder when $\mathsf{K} \gg 1$ and $\mathsf{M} < 1$ has also been studied. Even for rarefied flows f is not linear in V, but has an expansion of the form [9, 11]

$$f/f_0 = 1+\mathsf{K}^{-1}[b_1+b_1'\log \mathsf{M}]+ \dots , \tag{1.3}$$

where f_0 is the free molecular flow force. The log M term is the result of dynamical processes in which a particle, after colliding with the cylinder, makes a long excursion into the fluid, traveling several mean free paths before returning to the cylinder. Similar processes occur in the flow around a sphere, but they do not lead to divergences in F/F_0 as $\mathsf{M} \to 0$. In addition, there is a close relation between the K^{-1} expansion of f and the density expansion of the transport coefficients of a hypothetical two-dimensional gas [9]. We will discuss this relation further in Sect. 4.

Although the properties of slow gas flows at large Knudsen numbers seem to be well understood, as was mentioned in Problem (d) above it still remains to extend the theory to smaller Knudsen numbers and eventually to connect the results derived for both extremes.

In this paper we will discuss some of our recent work on problems (a)–(d) mentioned above. In Sect. 2 we will discuss our starting point, which is a way of writing the Boltzmann equation so that the boundary conditions satisfied by the distribution function at the object's surface are explicitly taken into account. In Sect. 3 we will use this Boltzmann equation to discuss the gas flow for $\mathsf{K} \ll 1$ and show that a generalization of the Chapman-Enskog normal solution method, when the boundary conditions are explicitly taken into account, leads to the hydrodynamic results for the gas flow. In Sect. 4 we will show how kinetic theory can explain the flow at high and low Knudsen numbers from a unified point of view — by showing that the dynamical events responsible for the force on the object are essentially the same in both cases. In Sect. 4 we will also discuss the difference between the flows around a sphere and the flows around a cylinder. In Sect. 5 we conclude with a brief discussion of the main results of this study. Here we will not give all the details of the relevant calculations, but merely summarize the main results. A more detailed description of these calculations will be published elsewhere.

2. The extended Boltzmann equation

We begin by considering the equation satisfied by $f(\mathbf{r}, \mathbf{v}, t)$, the single particle distribution function of the gas particles. We take $f(\mathbf{r}, \mathbf{v}, t)$ to be normalized so that $f(\mathbf{r}, \mathbf{v}, t)d\mathbf{r}d\mathbf{v}$ is the number of gas particles in $d\mathbf{r}$ about $\mathbf{r}$ and in $d\mathbf{v}$ about $\mathbf{v}$ at time t. In

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the absence of external forces, but in the presence of a macroscopic object, $f(\mathbf{r}, \mathbf{v}, t)$ can change in time through three different processes:

(i) Free streaming, where molecules at $(\mathbf{r}, \mathbf{v})$ at time t move to $(\mathbf{r}+\mathbf{v}dt, \mathbf{v})$ at time $t+dt$;

(ii) Collisions of gas molecules with each other. The effects of these collisions on $f(\mathbf{r}, \mathbf{v}, t)$ is taken to be given by the usual Boltzmann collision operator; and

(iii) Collisions of the gas molecules with the object. These collisions also cause $f(\mathbf{r}, \mathbf{v}, t)$ to change in time. Although these collisions are usually treated by formulating their effect on f as a boundary condition, there is no need to do so, and the result of the collisions can be incorporated directly into the Boltzmann equation. We will limit our discussions here to the case where the particles make specular or diffuse collisions with the object. Even more general collision mechanisms can be incorporated, but we have not yet attempted to discuss more general cases.

Taking into account the three above mentioned processes by which $f(\mathbf{r}, \mathbf{v}, t)$ may change in time, we are led to the extended Boltzmann equation

$$\frac{\partial f(\mathbf{r}, \mathbf{v}, t)}{\partial t} = \mathbf{v} \cdot \nabla f(\mathbf{r}, \mathbf{v}, t) + J(f,f) + \bar{T} f(\mathbf{r}, \mathbf{v}, t). \tag{2.1}$$

Here, the term $-\mathbf{v} \cdot \nabla f$ on the right-hand side of the Eq. (2.1) accounts for the change in f due to free streaming. $J(f,f)$ takes into account the molecule-molecule collisions and is given by

$$J(f,f) = \int d\mathbf{v}_1 \int_0^a b db \int_0^{2\pi} d\phi |\mathbf{v}-\mathbf{v}_1| [f(\mathbf{r}, \mathbf{v}_1', t) f(\mathbf{r}, \mathbf{v}_1', t) - f(\mathbf{r}, \mathbf{v}, t) f(\mathbf{r}, \mathbf{v}_1, t)], \tag{2.2}$$

where b is an impact parameter, ϕ the azimuthal angle, $\mathbf{v}'$ and $\mathbf{v}_1'$ the restituting velocities of two particles which lead to velocities $\mathbf{v}$ and $\mathbf{v}_1$ when the particles collide with impact parameter b and azimuthal angle $\phi+\pi$. The term $\bar{T}f$ describes the rate of change of F due to molecule-object collisions, and the precise form of $\bar{T}$ depends on the shape of the object and the interaction mechanism [11, 12, 13]. In the case when the object is a sphere, $\bar{T}f$ is given by

$$\bar{T}_{sp} f = R^2 \int d\hat{\sigma} |\mathbf{v} \cdot \hat{\sigma}| \{\theta(\mathbf{v} \cdot \hat{\sigma}) \delta(\mathbf{r} - R\hat{\sigma}) \times f(\mathbf{r}, \mathbf{v} - 2(\mathbf{v} \cdot \hat{\sigma})\hat{\sigma}, t) - \delta(\mathbf{r} - R\hat{\sigma}) \theta(-\mathbf{v} \cdot \hat{\sigma}) f(\mathbf{r}, \mathbf{v}, t)\} \tag{2.3}$$

and

$$\bar{T}_{di} f = R^2 \int d\hat{\sigma} |\mathbf{v} \cdot \hat{\sigma}| \{\theta(\mathbf{v} \cdot \hat{\sigma}) \phi_{0,w}(v) (2\pi\beta_w m)^{1/2} \times \delta(\mathbf{r} - R\hat{\sigma}) \int d\mathbf{v}' \theta(-\mathbf{v}' \cdot \hat{\sigma}) |\mathbf{v}' \cdot \hat{\sigma}| f(\mathbf{r}, \mathbf{v}', t) - \delta(\mathbf{r} - R\hat{\sigma}) \theta(-\mathbf{v} \cdot \sigma) f(\mathbf{r}, \mathbf{v}, t)\} \tag{2.4}$$

for specular and diffuse collisions, respectively. Here R is the radius of the sphere which is assumed to be centered at the origin [14], $\hat{\sigma}$ denotes a unit vector, $\theta(x)$ is the unit step function $\theta(x) = 1$ for $x \geqslant 0$, and is zero elsewhere. $\phi_{0,w}(v)$ is the equilibrium distribution

$$\phi_{0,w}(v) = \left(\frac{\beta_w m}{2\pi}\right)^{3/2} e^{-\frac{\beta_w m}{2} v^2}, \tag{2.5}$$

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where $T_w = (k_\beta \beta_w)^{-1}$ is the temperature of the macroscopic sphere. The action of the operators $\bar{T}_{sp}$ and $\bar{T}_{di}$ can be understood by referring to Fig. 2. Here we take the sphere to be centered at the origin, and the vertical axis to be in the direction of $\mathbf{v}$. To compute the rate of decrease of particles with velocity $\mathbf{v}$ through collisions with the sphere, we

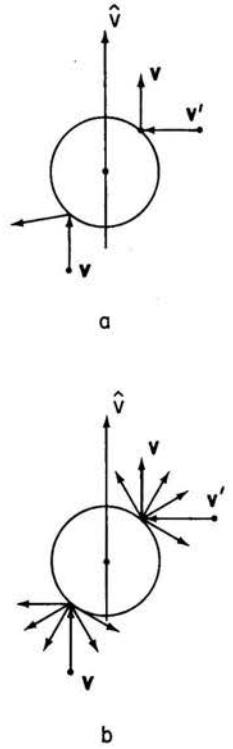

FIG. 2. Collisions of a fluid particle with the sphere. In Fig. 2a the fluid particles make specular collisions with the sphere. In Fig. 2b particles that collide with the sphere are re-emitted according to a Maxwell-Boltzmann distribution function.

note that such particles will collide with the sphere at point $\mathbf{r} = R\hat{\sigma}$, on the hemisphere $\mathbf{v} \cdot \hat{\sigma} < 0$. These collisions are accounted for by the second terms on the right-hand sides of Eqs. (2.3) and (2.4). The factor $|\mathbf{v} \cdot \hat{\sigma}|$ occurs because one needs the rate at which particles with velocity $\mathbf{v}$ strike the sphere. To compute the rate at which particles with velocity $\mathbf{v}$ are produced at the sphere, we need to consider the specular and diffuse mechanisms separately. In the case of specular reflection, particles with velocity $\mathbf{v}$ are produced at a point $\mathbf{r} = R\hat{\sigma}$, on the hemisphere $\mathbf{v} \cdot \hat{\sigma} > 0$, whenever a particle with velocity $\mathbf{v} - 2(\mathbf{v} \cdot \hat{\sigma})\hat{\sigma}$ strikes the sphere at the point. This accounts for the structure of the first term on the right-hand side of Eq. (2.3). In the case of diffuse reflection, we assume that whenever a particle hits the sphere it is absorbed and immediately re-emitted according to a Maxwell-Boltzmann distribution function, described by a temperature T_W, into direction pointing outward from the sphere. Therefore a molecule with velocity $\mathbf{v}$ is produced at a point $\mathbf{r} = R\hat{\sigma}$ on the hemisphere if a molecule with velocity $\mathbf{v}'$ strikes the sphere at this point ($\mathbf{v}' \cdot \hat{\sigma} < 0$) and is re-emitted with velocity $\mathbf{v}$. These facts, together with the requirement that the rate at which particles hit the sphere at the point $\mathbf{r} = R\hat{\sigma}$ be equal to the rate at which they leave, determine the structure of the first term on the right-hand side of Eq. (2.4).

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We now show that the extended Boltzmann equation (2.2) is equivalent to the ordinary Boltzmann equation, where the $\bar{T}f$ term does not appear, but is supplemented by the boundary conditions that are imposed on the distribution function at the surface of the sphere, in the case of specular or diffuse reflection. To do this, we look for solutions of Eq. (2.1) which vanish inside the sphere, since no particles should be located there. That is, we look for solution f of the form

$$f(\mathbf{r}, \mathbf{v}, t) = W(r)\tilde{f}(\mathbf{r}, \mathbf{v}, t), \tag{2.6}$$

where

$$W(r) = \begin{cases} 1 \text{ for } r \geqslant R, \\ 0 \text{ for } r < R \end{cases} \tag{2.7}$$

and where $\tilde{f}(\mathbf{r}, \mathbf{v}, t)$ is taken to be continuous, as a function of r, as $r \to R^{(+)}$. Substituting f, given by Eq. (2.6) into Eq. (2.1) and noting that

$$\mathbf{v} \cdot \nabla W(r) = \hat{} \cdot \mathbf{v}\delta(r-R), \tag{2.8}$$

where $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$ is a unit vector in the direction of $\mathbf{r}$, we obtain [15]

$$W(r)\left\{\frac{\partial \tilde{f}}{\partial t} + \mathbf{v} \cdot \nabla \tilde{f} - J(\tilde{f}, \tilde{f})\right\} = T\tilde{f}, \tag{2.9}$$

where

$$T\tilde{f} = \bar{T}\tilde{f} - \tilde{f}(\mathbf{v} \cdot \hat{\mathbf{r}})\,\delta(r-R). \tag{2.10}$$

Using Eqs. (2.3) and (2.4) we find that T_{sp} and T_{di} are given by

$$T_{sp}\tilde{f} = \theta(\mathbf{v} \cdot \hat{r})\,\delta(r-R)\,|\mathbf{v} \cdot \hat{\mathbf{r}}|\,[\tilde{f}(\mathbf{r}, \mathbf{v} - 2(\mathbf{v} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}, t) - \tilde{f}(\mathbf{r}, \mathbf{v}, t)] \tag{2.11}$$

and

$$T_{di}\tilde{f} = \theta(\mathbf{v} \cdot \hat{\mathbf{r}})\,\delta(r-R)\,|\mathbf{v} \cdot \hat{\mathbf{r}}|\,[\varphi_{0,w}(v)\,(2\pi\beta_w m)^{1/2} \times \int d\mathbf{v}' |\mathbf{v}' \cdot \hat{\mathbf{r}}| \theta(-\mathbf{v}' \cdot \hat{\mathbf{r}}) \tilde{f}(\mathbf{r}, \mathbf{v}', t) - \tilde{f}(\mathbf{r}, \mathbf{v}, t)]. \tag{2.12}$$

Since the right-hand side of Eq. (2.9) is proportional to $\delta(r-R)$, and $\tilde{f}$ is assumed to be continuous at $r = R$, a solution to this equation can only be found if the right and left-hand sides vanish identically. The vanishing of the lef-hand side requires that $\tilde{f}$ satisfy

$$\frac{\partial \tilde{f}}{\partial t} + \mathbf{v} \cdot \nabla \tilde{f} = J(\tilde{f}, \tilde{f}), \quad r > R \tag{2.13}$$

and the vanishing of the right-hand side requires, in the case of specular reflection, that

$$\tilde{f}(\mathbf{r}, \mathbf{v}, t) = \tilde{f}(\mathbf{r}, \mathbf{v} - 2(\mathbf{v} \cdot \hat{r})\hat{\mathbf{r}}, t), \quad \text{at } |\mathbf{r}| = R \quad \text{for } \mathbf{v} \cdot \hat{r} > 0 \tag{2.14}$$

and in the case of diffuse reflection that

$$\tilde{f}(\mathbf{r}, \mathbf{v}, t) = \varphi_{0,w}(2\pi\beta_w m)^{1/2} \int d\mathbf{v}' |\mathbf{v}' \cdot \hat{\mathbf{r}}| \theta(-\mathbf{v}' \cdot \hat{\mathbf{r}}) \tilde{f}(\mathbf{r}, \mathbf{v}', t), \quad \text{at } r = R \quad \text{for} \quad \mathbf{v} \cdot \hat{r} > 0. \tag{2.15}$$

The Eq. (2.13) together with the boundary conditions (2.14) and (2.15) form the usual starting point for the kinetic theory of gas flows [1] and we see that they are equi-

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valent to Eq. (2.1). We also see that from Eqs. (2.5), (2.9) and (2.11) or (2.12) the distribution function $f(\mathbf{r}, \mathbf{v}) = n\varphi_0(v)W(r)$ is the solution of Eq. (2.1) which corresponds to the case of total equilibrium. A similar discussion can be given for cylindrically shaped objects and so on, but we will not elaborate this here.

There are two principal advantages in using Eq. (2.1) as the starting point of our analysis.

(1) The boundary conditions are already incorporated into the equation, and this simplifies some of the mathematics.

(2) The treatment of the molecule-object collisions by means of a binary collision operator in the Boltzmann equation will greatly facilitate the dynamical analysis of the gas flows.

A third advantage is that the use of collision operators in this way allows us to treat the case where the macroscopic object can move, e.g., the case of Brownian motion, by methods similar to those employed here.

3. Normal solutions of the extended Boltzmann equation

As we discussed earlier, the Chapman-Enskog normal solution of the Boltzmann equation ignores the possible presence of boundaries, and hence provides a derivation of the hydrodynamic equations but not the boundary conditions needed to solve them. In this section we shall briefly consider the normal solutions of the extended Boltzmann equation (2.1), or equivalently Eq. (2.9), in order to discuss some of the features that arise when the boundary conditions are explicitly taken into account. We will discuss the flow around a sphere here and refer the reader to the discussion of more general flows given by CERCIGNANI [1, 2].

We first consider whether it is possible to apply the Chapman-Enskog method in order to obtain a normal solution to (2.9). This would require that outside the sphere $\tilde{f}$ be a Chapman-Enskog normal solution, while at the surface of the sphere $\tilde{f}$ should satisfy the condition $T\tilde{f} = 0$. From the work of Chapman and Enskog we know that the normal solution $f_{NS}(\mathbf{r}, \mathbf{v}, t)$ depends on time through the time variation of the local density $n(\mathbf{r}, t)$, the local velocity $\mathbf{u}(\mathbf{r}, t)$, and the local temperature $T(\mathbf{r}, t)$, and in addition is given by an expansion in powers of $\mathrm{K} = l/R$, the Knudsen number, in the form

$$f_{NS}(\mathbf{r}, \mathbf{v}, t) = f_0(\mathbf{r}, \mathbf{v}|n, \mathbf{u}, T) + f_1(\mathbf{r}, \mathbf{v}|n, \mathbf{u}, T) + \ldots, \tag{3.1}$$

where f_0 is a local equilibrium distribution given by

$$f_0(\mathbf{r}, \mathbf{v}|n, \mathbf{u}, T) = n(\mathbf{r}, t)\left(\frac{\beta(\mathbf{r}, t)m}{2\pi}\right)^{3/2} \exp - \frac{\beta(\mathbf{r}, t)m}{2} C^2(\mathbf{r}, t), \tag{3.2}$$

where $\beta(\mathbf{r}, t) = (k_\beta T(\mathbf{r}, t))^{-1}$ and $\mathbf{C}(\mathbf{r}, t) = \mathbf{v} - \mathbf{u}(\mathbf{r}, t)$, f_1 is proportional to the gradients of the hydrodynamic variables and is given by

$$f_1(\mathbf{r}, \mathbf{v}|n, \mathbf{u}, T) = f_0\left[A(C^2)\left(\frac{\beta(\mathbf{r}, t)m}{2} C^2 - 5/2\right) C \cdot \nabla \log T(\mathbf{r}, t) + B(C^2)\left(\mathbf{CC} - \frac{C^2}{3} I\right) : \nabla \mathbf{u}(\mathbf{r}, t).\right. \tag{3.3}$$

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Here I is the unit dyadic tensor, and the functions $A(C^2)$ and $B(C^2)$ are determined by solving an inhomogeneous, linearized Boltzmann equation [3]. This normal solution, when inserted into the equations expressing the conservation of particle number, momentum, and energy, then leads to the result that n, $\mathbf{u}$, T satisfy the hydrodynamic equations. For example, keeping only f_0 in Eq. (3.1) one obtains the Euler equations, for f_0 and f_1 one obtains the Navier-Stokes equations, and so on. However, to complete the discussion one has to show that f_{NS} satisfies $Tf_{NS} = 0$. Since f_{NS} is expanded in powers of l/R, this requirement implies that

$$Tf_i(\mathbf{r}, \mathbf{v}|n, \mathbf{u}, T) = 0, \text{ for } i = 0, 1, 2 \ldots . \tag{3.4}$$

If one restricts oneself to the leading orders, boundary conditions can be obtained from Eq. (3.4) for specular and for diffuse reflection, respectively, and are the familiar hydrodynamic boundary conditions, given in Table 1. The boundary conditions for specular

Table 1. Boundary conditions on the hydrodynamic variables at the surface of sphere $r = R$.

	Specular Reflection	Diffuse Reflection
	$\mathbf{u}(\mathbf{r}) \cdot \hat{\mathbf{r}} = 0$	$\mathbf{u}(\mathbf{r}) = 0$
	$(\hat{\mathbf{r}}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}\hat{\mathbf{r}})$: $\nabla\mathbf{u} = 0$	$T(\mathbf{r}) = T_w$
	$(\hat{\mathbf{r}}\hat{\boldsymbol{\Theta}} + \hat{\boldsymbol{\Theta}}\hat{\mathbf{r}})$: $\nabla\mathbf{u} = 0$	
where		
	$\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{r}} \cdot \hat{\boldsymbol{\Theta}} = 0$	
	$\hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\Theta}} \cdot \hat{\boldsymbol{\Theta}} = 1$	
	$\hat{\mathbf{r}} \cdot \nabla_r T(\mathbf{r}) = 0$	

reflection are derived from Eq. (3.4) using $i = 0, 1$, and for diffuse reflection, using only $i = 0$.

These boundary conditions lead to expressions for the force on the sphere given by

$$F_{sp} = 4\pi\eta RV, \quad F_{di} = 6\pi\eta RV \tag{3.4'}$$

for specular and diffuse reflection, respectively. Here η is the shear viscosity of the gas. One of the interesting consequences of this analysis appears when one introduces a reflection mechanism which is a linear combination of specular and diffuse reflection with accomodation coefficient α by [1, 2, 16]

$$\bar{T}_\alpha = \alpha\bar{T}_{di} + (1-\alpha)\bar{T}_{sp}. \tag{3.5}$$

When α is of order K, i.e. $\alpha \sim l/R$, $\bar{T}_\alpha$ leads to a boundary condition where the tangential stress on the sphere is proportional to the tangential fluid velocity at the surface, and the corresponding slip coefficient ζ_s is

$$\zeta_s = 2/\alpha. \tag{3.6}$$

The result has been obtained before by more elementary methods [16]. The force on the sphere is then given by

$$F = 6\pi\eta RV\left(\frac{2R+2\zeta_s\lambda}{2R+3\zeta_s\lambda}\right), \tag{3.7}$$

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where λ is a length on the order of a mean free path, given by $\lambda = \eta(\beta m\pi/2)^{1/2}(nm)^{-1}$. The Eq. (3.7) holds only as long as α is $0(l/k)$, but for all α the product $\alpha\zeta_s$ is of order 1. Therefore when α is 0(1), stick boundary conditions are correct up to terms of order l/R, and the force on the sphere is given by the usual form of Stokes Law, Eq. $(3.4')_2$.

If one attempts to use (3.4) to derive higher order boundary conditions on the hydrodynamic variables for use in solving the Burnett, super-Burnett, etc., hydrodynamic equations, one is led immediately to serious inconsistencies with the lower order boundary conditions, and other difficulties [17] which force one to conclude that after some order in l/R the normal solution is no longer a solution of the extended Boltzmann equation. This is not surprising since it is well known, from other kinetic theory calculations, that there are boundary layers close to the surface of the sphere in which the distribution function changes over distances of the order of a mean free path, and hence cannot be described by a normal solution of the Boltzmann equation. Therefore we must look for generalized normal solutions of the extended Boltzmann equation which approach the Chapman-Enskog solution far from the sphere but include the effects of the boundary layer extending over a few mean free paths in the vicinity of the surface of the sphere.

For simple model gases and for simple gas flows — for example, the flow of a B.G.K. gas past a fixed wall [1] — it is possible to find the generalized normal solution explicitly, and to exhibit the boundary layer effects. However, even for this simple case a number of problems remain to be solved. For example, in spite of the fact that the distribution function is known, the role of the higher order hydrodynamic equations is not clear, nor is it clear how to derive the boundary conditions that must be used to obtain their solution.

In the discussion of dynamical events taking place in the gas (in the next section) we will sketch the derivation of an equation [Eq. (4.22)] which might be a convenient starting point for the derivation of the generalized normal solutions of the stationary, linearized Boltzmann equation. These solutions, in turn, lead to the stationary, linearized hydrodynamic equations with the boundary layers taken into account. However, we have not yet made a detailed study of these generalized normal solutions.

We may conclude, therefore, that a generalization of the Chapman-Enskog normal solution method to this case where the boundary is taken into account does lead to Stokes Law for the flow around a sphere, but the structure of the boundary layer and the role of higher order hydrodynamic equations needs considerably more investigation.

4. From rarefied gas dynamics to hydrodynamics

When the mean free path l is much larger than the characteristic size, R, of the object, the equations of hydrodynamics cannot be used to describe the gas flow, since it is assumed in their derivation that l/R is small. Instead, when $l/R \gg 1$ one may regard the molecule-object collisions as a perturbation to the free molecular flow of the gas. In this section we will show how the equations of rarefied gas dynamics are obtained from the extended Boltzmann equation. We will also show how these equations may be used to obtain an analysis of hydrodynamic flows in terms of dynamical events taking place in the gas, and

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to obtain the correct form of Stokes' Law, and Lamb's formula for the force on a sphere and cylinder, respectively. We will first consider the force on a sphere of radius R, and then discuss the modifications needed to derive expressions for the force on a cylinder.

By considering the momentum transferred to the sphere by the gas particles, we easily find that the force on the sphere is given by the formula

$$\mathbf{F} = -\int d\mathbf{r} \int d\mathbf{v} m\mathbf{v}\bar{T}f(\mathbf{r}, \mathbf{v}, t). \tag{4.1}$$

To obtain an expression for f, we consider the case where the Mach number $\mathsf{M} \ll 1$, and we suppose that the disturbance in the gas caused by the presence of the sphere is small enough that we may expand the solution of the Eq. (2.1) about the equilibrium solution, and keep only terms linear in the deviation from equilibrium. We assume that infinitely far away from the sphere the distribution function of the gas is

$$f(\mathbf{r}, \mathbf{v}) = n\left(\frac{\beta m}{2\pi}\right)^{3/2} \exp -\frac{\beta m}{2}(\mathbf{v}-\mathbf{V})^2 \approx n\varphi_0(v)\,(1+\beta m\mathbf{v}\cdot\mathbf{V}), \text{ as } r \to \infty \tag{4.2}$$

and suppose that the temperature of gas at $r \to \infty$ is equal to that of the sphere, in the case of diffuse reflection. Since we are interested in the steady state force on the sphere, we look for stationary solutions of the Eq. (2.1) of the form

$$f(\mathbf{r}, \mathbf{v}) = nW(r)\varphi_0(v) + n\Psi(\mathbf{r}, \mathbf{v})\varphi_0 \tag{4.3}$$

and the equation for, Ψ is [18]

$$(\mathbf{v}\cdot\nabla - L - \bar{T})\Psi(\mathbf{r}, \mathbf{v})\varphi_0(v) = 0, \tag{4.4}$$

where the linearized Boltzmann collision operator is given by

$$L\Psi\varphi_0 = n\int d\mathbf{v}_1 \int_0^a bdb \int_0^{2\pi} d\varphi |\mathbf{v}-\mathbf{v}_1|\varphi_0(v)\varphi_0(v_1)[\Psi(\mathbf{r}, \mathbf{v}_1') + \Psi(\mathbf{r}, \mathbf{v}') - \Psi(\mathbf{r}, \mathbf{v}) - \Psi(\mathbf{r}, \mathbf{v}_1)]. \tag{4.5}$$

In deriving Eq. (4.3) we have used the fact that $nW(r)\varphi_0(v)$ is the equilibrium solution of Eq. (2.1). We separate the distribution function into its asymptotic part and a part $\delta\psi$ which vanishes far from the sphere, as

$$\Psi(\mathbf{r}, \mathbf{v}) = \beta m\mathbf{v}\cdot\mathbf{V} + \delta\psi(\mathbf{r}, \mathbf{v}) \tag{4.6}$$

to write the Eq. (4.4) as

$$(\mathbf{v}\cdot\nabla - L - \bar{T})\,\delta\psi = \bar{T}\beta m\mathbf{v}\cdot\mathbf{V}, \tag{4.7}$$

where we have used the fact that both $\mathbf{v}\cdot\nabla$ and L vanish when acting on $\mathbf{v}\cdot\mathbf{V}\varphi_0$. The Eq. (4.6) leads to an expression for $\delta\psi$ which is linear in $\mathbf{V}$. We may write the solution of Eq. (4.6) formally as

$$\delta\psi(\mathbf{r}, \mathbf{v})\varphi_0(v) = (\mathbf{v}\cdot\nabla - L - \bar{T})^{-1}T\beta m\mathbf{v}\cdot\mathbf{V}\varphi_0. \tag{4.8}$$

Further, by iterating the inverse operator about the operator $(\mathbf{v}\cdot\nabla - L)^{-1}$, called the Boltzmann propagator, we obtain an expression for $\delta\psi\varphi_0(v)$ as

$$\delta\psi(\mathbf{r}, \mathbf{v})\varphi_0(\mathbf{v}) = \{(\mathbf{v}\cdot\nabla - L)^{-1}\bar{T} + (\mathbf{v}\cdot\nabla - L)^{-1}\bar{T}(\mathbf{v}\cdot\nabla - L)^{-1} + (\mathbf{v}\cdot\nabla - L)^{-1}\bar{T}(\mathbf{v}\cdot\nabla - L)^{-1}\bar{T}(\mathbf{v}\cdot\nabla - L)^{-1} + \ldots\}\beta m\mathbf{v}\cdot\mathbf{V}\varphi_0. \tag{4.9}$$

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Using Eqs. (4.1), (4.3), (4.6) and (4.9), we obtain an expression for **F**

$$\mathbf{F} = -n\int d\mathbf{r}\int d\mathbf{v}\, m\mathbf{v}\{\bar{T}+\bar{T}(\mathbf{v}\cdot\nabla-L)^{-1}\bar{T}+\ \ldots\}\beta m\mathbf{v}\cdot\mathbf{V}\varphi_0(v), \tag{4.10}$$

where we have used the fact that the equilibrium distribution function $nW(r)\varphi_0(v)$ does not give any contribution to the force on the sphere. The first term in the expansion represents the *free molecular flow* force on the sphere. The second term contains all dynamical processes in which there are two collisions between gas particles and the sphere, and an arbitrary number of intermediate collisions between the gas molecules. The third term contains processes with three collisions between the gas molecules and the sphere, separated by intermediate collisions among the gas molecules, and so on. An expansion of **F** in powers of the inverse Knudsen number $K^{-1} = R/l$ can be obtained from Eq. (4.10) by expanding the Boltzmann propagators in powers of the free particle propagator $(\mathbf{v}\cdot\nabla)^{-1}$. This leads to the following expansion for **F**

$$\begin{aligned}\mathbf{F} = -n\int d\mathbf{r}\int d\mathbf{v}\, m\mathbf{v}\{&\bar{T}+\bar{T}(\mathbf{v}\cdot\nabla)^{-1}L(\mathbf{v}\cdot\nabla)^{-1}T\\ &+\bar{T}(\mathbf{v}\cdot\nabla)^{-1}L(\mathbf{v}\cdot\nabla)^{-1}L(\mathbf{v}\cdot\nabla)^{-1}\bar{T}+\bar{T}(\mathbf{v}\cdot\nabla)^{-1}L(\mathbf{v}\cdot\nabla)^{-1}\\ &\times\bar{T}(\mathbf{v}\cdot\nabla)^{-1}L(\mathbf{v}\cdot\nabla)^{-1}\bar{T}+\ \ldots\}\beta m\mathbf{v}\cdot\mathbf{V}\varphi_0,\end{aligned} \tag{4.11}$$

where we have used the identity $\bar{T}(\mathbf{v}\cdot\nabla)^{-1}\bar{T} = 0$, based on the fact that a particle cannot hit the sphere more than once without the intervention of at least one other gas particle. The first term on the right in Eq. (4.11), the free molecular flow contribution, is F_0 in Eq. (1.1) and represents the contribution to the force from processes where particles whose momentum distribution is $\varphi_0(1+\beta m\mathbf{v}\cdot\mathbf{V})$ collide once with the sphere. This process is illustrated in Fig. 1a. The force due to these collisions is proportional to the cross-sectional area of the sphere, πR^2, and to the velocity, **V**. The second term represents contributions from dynamical processes taking place between two gas particles and the sphere where one of the particles with momentum distribution $\varphi_0(1+\beta m\mathbf{v}\cdot\mathbf{V})$ collides with the sphere, then suffers a collision with the other gas particle, and then one of the two gas particles involved in the collision hits the sphere again. Events of this type are sketched in Fig. 1b,c,d. These events give corrections of $0(R/l)$ to the free molecular flow force, and determine the coefficient a_1 in Eq. (1.1). For a B.G.K. model gas, the coefficient has been explicitly computed by WILLIS, by WANG, and by KAN, and for a hard sphere gas a_1 has been computed by SENGERS, KUPERMAN and WANG, and for Maxwell molecules, by LIU, *et al* [10]. The third and fourth terms on the right in Eq. (4.11) take into account the dynamical processes between three gas particles and the sphere in which there are two molecule-molecule collisions and two or three molecule-object collisions, respectively. Typical events which contribute to the third and fourth terms in Eq. (4.11) are illustrated in Fig. 1e,f, respectively. While the third term on the right in Eq. (4.11) formally gives corrections of order K^{-2} to F_0, its contribution to F is actually divergent [11]. Moreover, almost all of the terms in Eq. (4.11) are divergent. This divergence is analogous to the divergence that appears in the order $(na^3)^2$ term in the virial expansion of the transport coefficients of a moderately dense gas, and is due to collision processes where the particles travel arbitrarily long distances between collisions. Actually, because of the presence of all the other particles in the gas, a particle travels only a mean

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free path or so between collisions. The individual terms in the expansion given in Eq. (4.11) fail to take these other collisions into account, and one must perform a resummation of the terms in the K^{-1} expansion in order to include them. From these arguments one can conclude that the expansion of the Boltzmann propagator $(\mathbf{v}\cdot\nabla - L)^{-1}$ in Eq. (4.10) in powers of the free particle propagators should not have been made. In view of the divergence difficulties of Eq. (4.11), workers in rarefied gas dynamics usually expand the Boltzmann propagators, using $L = L_N + L_I$, as

$$(\mathbf{v}\cdot\nabla - L)^{-1} = (\mathbf{v}\cdot\nabla - L_N)^{-1} + (\mathbf{v}\cdot\nabla - L_N)^{-1} L_I (\mathbf{v}\cdot\nabla - L_N)^{-1} + \ldots, \tag{4.12}$$

where for an arbitrary function of velocity, $g(\mathbf{v})$, $L_N\varphi_0(v)g(\mathbf{v})$ is given by

$$L_N\varphi_0(v)g(\mathbf{v}) = -nk(v)\varphi_0(v)g(\mathbf{v}) \tag{4.13}$$

with

$$k(v) = \pi a^2 \int d\mathbf{v}_1 |\mathbf{v}-\mathbf{v}_1| \varphi_0(v_1). \tag{4.14}$$

The operator L_N takes into account a "collisional damping" which restricts the length of the trajectory of a particle in the gas to a mean free path or so. When the expansion given by Eq. (4.12) is inserted into Eq. (4.10), an expansion of F is obtained where every term appears to be finite, and the resulting expansion of F in terms of K^{-1} is given by Eq. (1.1). The resummation does not affect the coefficient of order K^{-1}, but replaces a divergent term of order K^{-2} by a finite term of order $\mathsf{K}^{-2}\log\mathsf{K}^{-1}$. For a B.G.K. gas, the coefficient a_2' has been computed by KAN [10] and for this model one has that

$$\mathbf{F} = \mathbf{F}_0(1 - 0.4293\mathsf{K}^{-1} + 0.0672\mathsf{K}^{-2}\log\mathsf{K}^{-1} + \ldots), \tag{4.15}$$

where

$$\mathbf{F}_0 = \frac{2\mathbf{V}}{3}\frac{(nm\pi R^2)(8+\pi)}{(2\pi\beta m)^{1/2}}; \quad \mathsf{K}^{-1} = \frac{nR}{\eta_0}\left(\frac{m}{2\beta}\right)^{1/2},$$

where η_0 is the coefficient of shear viscosity for the B.G.K. gas, in the case that the gas molecules are reflected diffusely from the sphere. However, there is as yet no proof that the expansion given by (1.1) is convergent.

At this point an interesting question arises: since it is possible to associate specific dynamical events with each term in the expansion of the right-hand side of Eq. (4.11), is it possible to derive Stokes' Law by using this expansion, and thus learn which dynamical events are important in the hydrodynamic regime? In fact, we have shown that this is possible, and to do it we proceed as follows.

To derive Stokes' Law from Eq. (4.11), we do not iterate the Boltzmann propagators about the free particle propagator, but instead separate $(\mathbf{v}\cdot\nabla - L)^{-1}$ into a hydrodynamic part and a nonhydrodynamic part, as

$$(\mathbf{v}\cdot\nabla - L)^{-1} = P(\mathbf{v}\cdot\nabla - L)^{-1}P + P_\perp(\mathbf{v}\cdot\nabla - L)^{-1}P_\perp. \tag{4.16}$$

Here P is a projection operator which projects onto the space of the (linearized) normal solution of the Boltzmann equation, given by

$$Pg(\mathbf{r},\mathbf{v})\varphi_0 = \Phi_n(\mathbf{r},\mathbf{v})\varphi_0(v)a_n(\mathbf{r}) + \boldsymbol{\Phi}_v(\mathbf{r},\mathbf{v})\cdot\mathbf{a}_v(\mathbf{r})\varphi_0(v) + \Phi_T(\mathbf{r},\mathbf{v})a_T(\mathbf{r})\varphi_0(v), \tag{4.17}$$

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where

$$\Phi_n(\mathbf{r}, \mathbf{v}) = 1,$$

$$\Phi_v(\mathbf{r}, \mathbf{v}) = (\beta m)^{1/2}\left\{\mathbf{v} + \frac{B(v^2)}{\beta m}\left(\mathbf{v}\mathbf{v} - \frac{v^2}{3} I\right)\cdot\nabla + 0(\nabla^2)\right\}, \tag{4.18}$$

$$\Phi_T(\mathbf{r}, \mathbf{v}) = \left(\frac{2}{3}\right)^{1/2}\left\{\left(\frac{\beta m}{2} v^2 - 3/2\right) + A(v)\left(\frac{\beta m}{2} v^2 - \frac{5}{2}\right)\mathbf{v}\cdot\nabla + 0(\nabla^2)\right\},$$

$$a_i(\mathbf{r}) = \int d\mathbf{v}\Phi_i(\mathbf{r}, \mathbf{v}) g(\mathbf{r}, \mathbf{v})\varphi_0(v), \quad i = n, \mathbf{v}, T$$

and $A(v^2)$, $B(v^2)$ are determined as the solutions of the linearized Boltzmann equations

$$LA(v^2)\left(\frac{\beta m}{2} v^2 - \frac{5}{2}\right)\mathbf{v}\varphi_0 = \left(\frac{\beta m}{2} v^2 - \frac{5}{2}\right)\mathbf{v}\varphi_0 \tag{4.19}$$

and

$$LB(v^2)\left(\mathbf{v}\mathbf{v} - \frac{v^2}{3} I\right)\varphi_0 = \beta m\left(\mathbf{v}\mathbf{v} - \frac{v^2}{3} I\right), \tag{4.20}$$

respectively. Also $g(\mathbf{r}, \mathbf{v})$ is an arbitrary function of $\mathbf{r}$ and $\mathbf{v}$. $P_\perp$ projects on the orthogonal space, and both P and $P_\perp$ commute with $(\mathbf{v}\cdot\nabla - L)^{-1}$. Similarly we may separate $\delta\psi$ into a hydrodynamic part $P\delta\psi$ and an orthogonal part $P_\perp\delta\psi$. By inserting Eq. (4.16) into Eq. (4.9) and collecting some terms we obtain an expansion for $P\delta\psi$ given by

$$P\delta\psi\varphi_0 = \{P(\mathbf{v}\cdot\nabla - L)^{-1}P\tilde{T} + P(\mathbf{v}\cdot\nabla - L)^{-1}P\tilde{T}P(\mathbf{v}\cdot\nabla - L)^{-1}P\tilde{T} + \ldots\}\beta m\mathbf{v}\cdot\mathbf{V}\varphi_0 \tag{4.21}$$

or

$$P\delta\psi\varphi_0 = [P(\mathbf{v}\cdot\nabla - L)P - \tilde{T}]^{-1}P\tilde{T}\beta m\mathbf{v}\cdot\mathbf{V}\varphi_0, \tag{4.22}$$

where

$$\tilde{T} = P\bar{T}P + P\bar{T}P_\perp[P_\perp(\mathbf{v}\cdot\nabla - L - \bar{T})P_\perp]^{-1}P_\perp\bar{T}P \tag{4.23}$$

or

$$\tilde{T} = P\bar{T}P + P\bar{T}\{P_\perp(\mathbf{v}\cdot\nabla - L)^{-1}P_\perp + P_\perp(\mathbf{v}\cdot\nabla - L)^{-1}P_\perp\bar{T}P_\perp(\mathbf{v}\cdot\nabla - L)^{-1}P_\perp + \ldots\}P_\perp TP. \tag{4.24}$$

The orthogonal part $P_\perp\delta\psi\varphi_0$ can also be expressed in terms of $\tilde{T}$ and the projected propagators, but we omit the details here. The first term in the right side of Eq. (4.24) for the collision operator $\tilde{T}$ describes dynamical processes in which the fluid particle collides once with the sphere [19]. The remaining terms in the expression for $\tilde{T}$ describe processes where the particles suffer any number of collisions with the sphere, but the particles travel only a mean free path or so between such collisions, since the propagator $P_\perp(\mathbf{v}\cdot\nabla - L)^{-1}P_\perp$ describes processes where the particle travels only a distance of the order of mean free path. On the other hand, the hydrodynamic propagator $P(\mathbf{v}\cdot\nabla - L)^{-1}P$ describes dynamical processes in which fluid particles travel over distances of several mean free paths and suffer many collisions with other fluid particles. Thus, the dynamical events which contribute to the terms in the expansion (4.21) can be considered as combinations of collisions

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of fluid particles with the sphere, described by $\tilde{T}$, connected by long excursions of fluid particles into the fluid, described by $P(v\cdot\nabla - L)^{-1}P$. The sum in the curly brackets in Eq. (4.21) can now be written as

$$P\delta\psi\varphi_0(v) = \sum_{n=1}^{\infty} P\delta\psi^{(n)}(\mathbf{r},\mathbf{v})\varphi_0(v) \tag{4.25}$$

with

$$P\delta\psi^{(n)}(\mathbf{r},\mathbf{v})\varphi_0 = [P(\mathbf{v}\cdot\nabla - L)^{-1}P\tilde{T}]^n \beta m\mathbf{v}\cdot\mathbf{V}\varphi_0(v). \tag{4.26}$$

To leading order in the Knudsen number these equations can be solved. For the case of diffuse reflection $P\delta\psi^{(n)}\varphi_0$ is given to leading order by combination of two normal solutions as

$$P\delta\psi^{(n)}\varphi_0 = \sum_{i=1}^{2} a_i^{(n)} u_i(\mathbf{r},\mathbf{v})\varphi_0(v) \tag{4.27}$$

where

$$u_1(\mathbf{r},\mathbf{v}) = \Phi_v(\mathbf{r},\mathbf{v})\cdot\mathbf{u}_1(\mathbf{r}), \tag{4.28}$$

$$u_2(\mathbf{r},\mathbf{v}) = \Phi_v(\mathbf{r},\mathbf{v})\cdot\mathbf{u}_2(\mathbf{r}) \tag{4.29}$$

with Φ_v given by the Eq. (4.18), and flow fields $\mathbf{u}_1$ and $\mathbf{u}_2(r)$ are

$$\mathbf{u}_1(r) = \begin{cases} (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\left(\frac{1}{3}\frac{R}{r} - \frac{1}{15}\left(\frac{R}{r}\right)^3\right) + (\hat{\mathbf{z}} - (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\left(\frac{1}{6}\frac{R}{r} + \frac{1}{30}\left(\frac{R}{r}\right)^3\right) & \text{for } r > R, \\ (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\left(\frac{1}{3}\frac{r}{R} - \frac{1}{15}\left(\frac{r}{R}\right)^3\right) + (\hat{\mathbf{z}} - (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\left(\frac{1}{3}\frac{r}{R} - \frac{2}{15}\left(\frac{r}{R}\right)^3\right) & \text{for } r < R; \end{cases} \tag{4.30}$$

$$\mathbf{u}_2(r) = \begin{cases} (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\left(\frac{2}{3}\frac{R}{r} - \frac{4}{15}\left(\frac{R}{r}\right)^3\right) + (\hat{\mathbf{z}} - (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\left(\frac{1}{3}\frac{R}{r} + \frac{2}{15}\left(\frac{R}{r}\right)^3\right) & \text{for } r > R, \\ (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\left(\frac{1}{3}\frac{r}{R} + \frac{1}{15}\left(\frac{r}{R}\right)^3\right) + (\hat{\mathbf{z}} - (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\left(\frac{1}{3}\frac{r}{R} + \frac{2}{15}\left(\frac{r}{R}\right)^3\right) & \text{for } r < R, \end{cases} \tag{4.31}$$

where $\hat{\mathbf{z}}$ is a unit vector in the direction of $\mathbf{V}$. The successive values of $a_i^{(n+1)}$ are related by the equation

$$a_i^{(n+1)} = \sum_j \mathsf{M}_{ij} a_j^{(n)}, \tag{4.32}$$

where M is the 2×2 matrix

$$\mathsf{M} = \begin{bmatrix} -\frac{4}{15}\alpha\frac{R}{\lambda} & -\frac{2}{5}\alpha\frac{R}{\lambda} \\ -\frac{1}{5}\beta\frac{R}{\lambda} & -\frac{7}{15}\beta\frac{R}{\lambda} \end{bmatrix}, \tag{4.33}$$

where $\lambda = \eta_0(nm)^{-1}(\beta\pi m/2)^{1/2}$, and α and β are positive. They are functions of R/λ which appear to be of order unity for all R/λ. To calculate M, we start from the relation

$$P\delta\psi^{(n+1)}(\mathbf{r},\mathbf{v})\varphi_0 = P(\mathbf{v}\cdot\nabla - L)^{-1}P\tilde{T}P\delta\psi^{(n)}\varphi_0 \tag{4.34}$$

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which we write as

$$P(\mathbf{v}\cdot\nabla - L)P\delta\psi^{(n+1)}\varphi_0 = P\tilde{T}P\delta\psi^{(n)}\varphi_0 \tag{4.35}$$

and then substitute Eq. (4.27) for $\delta\psi^{(n)}$ on the right-hand side. One can show that

$$P\tilde{T}P\delta\psi^{(n)}\varphi_0 = -\left(\frac{4}{15}a_1^{(n)} + \frac{2}{5}a_2^{(n)}\right)\alpha(\boldsymbol{\Phi}_v\cdot\hat{\mathbf{r}})\,(\hat{\mathbf{r}}\cdot\hat{\mathbf{z}})\,\delta(r-R)\varphi_0 - \left(\frac{1}{5}a_1^{(n)} + \frac{7}{15}a_2^{(n)}\right)\beta\boldsymbol{\Phi}_v\cdot(\hat{\mathbf{z}} - (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\,\delta(r-R)\varphi_0. \tag{4.36}$$

It is then possible to solve Eq. (4.35), to write the solution for $\delta\psi^{(n+1)}$ also in the form of Eq. (4.27), and to obtain M by relating $a_i^{(n+1)}$ to $a_i^{(n)}$. The factors α and β are determined by the structure of the boundary layer and their exact values are not known.

From Eqs. (4.25), (4.27) and (4.32), it follows that $P\delta\psi\varphi_0$ is

$$P\delta\psi\varphi_0 = \sum_{i,j}[\mathbf{1}-\mathsf{M}]_{ij}^{-1}a_j^{(1)}u_j\varphi_0, \tag{4.37}$$

where $\mathbf{1}$ is the 2×2 unit matrix. As can be seen from Eq. (4.33) for M, the summation of $P\delta\psi^{(n)}$ leading to Eq. (4.37) is a summation of a power series in R/l. In the hydrodynamic limit, R/l is a large quantity, and in order to apply the summation to the hydrodynamic limit, we regard to Eq. (4.37) as the analytic continuation of the result obtained by summing the terms in Eq. (4.27) inside their radius of convergence.

The results obtained from Eq. (4.37) in the limit $R/l \gg 1$ agree completely with those obtained by means of the normal solution. To leading order (now in l/R) the constants α and β drop out, indicating that to leading order only the reflection mechanism of the particles by the sphere is important, but not the structure of the boundary layer. In fact the same result is obtained if $\tilde{T}$ is replaced by the first term in Eqs. (4.23), (4.24), $P\overline{T}P$. This shows that the dynamical events which are responsible for Stokes' Law are those in which the gas particles make long excursions into the fluid between collisions with the sphere.

The force on the sphere can be shown to be given by

$$\mathbf{F} = -n\int d\mathbf{r}\int d\mathbf{v}m\mathbf{v}\tilde{T}[\beta m\mathbf{v}\cdot\mathbf{V}\varphi_0 + P\delta\psi\varphi_0] \tag{4.38}$$

and the evaluation of the force to leading order in l/R leads to Stokes' Law, $\mathbf{F} = 6\pi\eta R\mathbf{V}$. In a similar way the case of specular reflection and the intermediate case of a linear combination of specular and diffuse reflection lead to the results for $\mathbf{F}$ given by Eqs. $(3.4)_1$ and (3.7), respectively.

If we attempt to apply the preceeding analysis to compute the force per unit length on a cylinder, $\mathbf{f}$, for both large and small K, serious difficulties arise almost at once. If we assume that, like the force on a sphere, $\mathbf{f}$ is linear in $\mathbf{V}$, and then try to expand $\mathbf{f}$, for large K, as a power series in K^{-1}, using an expression similar to Eq. (4.11) for the sphere, we find that the first correction to the free molecular flow force, $\mathbf{f}_0$, is divergent, as are all higher terms in the expansion of $\mathbf{f}$ in powers of K^{-1} [11]. This divergence forces us to return to the analog of Eqs. (4.10), (4.11) for the cylinder

$$\mathbf{f} = -\lim_{L\to\infty} n\frac{1}{L}\int d\mathbf{r}\int d\mathbf{v}m\mathbf{v}\{\overline{T}_{cyl} + \overline{T}_{cyl}(\mathbf{v}\cdot\nabla - L)^{-1}\overline{T}_{cyl} + \ldots\}\beta m\mathbf{v}\cdot\mathbf{V}\varphi_0 \tag{4.39}$$

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where L is the length of the cylinder and $\overline{T}_{cyl}$ is the collision operator for molecule-cylinder collisions. Unlike the case of the sphere, the second term and all higher terms in the expansion are divergent even at large Knudsen numbers. These divergences come from the contribution of the hydrodynamic part of the $(\mathbf{v}\cdot\nabla - L)^{-1}$ operator and are due to collision processes where the particles travel several mean free paths between collisions; they are similar to the divergences that appear in the "renormalized" theory of transport coefficients for a hypothetical two-dimensional gas. Unlike the divergence in the K^{-1} expansion, the divergences in Eq. (4.39) cannot be removed by further summation of the terms in the expansion. Instead, the difficulty can be traced to the assumption that the distribution function and the force are linear in $\mathbf{V}$. This difficulty can be overcome by linearizing the distribution function about the asymptotically correct distribution function $n\varphi_0(\mathbf{v}, \mathbf{V})$ given by

$$n\varphi_0(\mathbf{v}, \mathbf{V}) = n\left(\frac{\beta m}{2\pi}\right)^{3/2} e^{-\frac{\beta m}{2}(\mathbf{v}-\mathbf{V})^2}$$

and retaining higher terms in $\mathbf{V}$. For the case of high Knudsen numbers, this linearization leads to an expansion for $\mathbf{f}$ as [9a]

$$f/f_0 = 1 + \mathsf{K}^{-1}[b_1 + b_1' \log \mathsf{M}] + \ldots \tag{4.40}$$

where M is the Mach number V/c, and non-linear terms in V appear. To treat the case where $\mathsf{K} \ll 1$, one can use the normal solution method outlined in Sect. 3, but the Ossen corrections must be used to solve the resulting hydrodynamic equations; or one can use an iteration procedure similar to that outlined here. In the latter procedure one sums a geometric series in powers of $[(R/l)\log(V/c)]$. The resulting expression for $\mathbf{f}$, valid for $K \ll 1$, is

$$\mathbf{f} = \frac{4\pi\eta\mathbf{V}}{\frac{1}{2} - C - \log\frac{1}{4}\mathrm{Re}}, \tag{4.41}$$

where C is Euler's constant and the Reynolds number Re is

$$\mathrm{Re} = \frac{RnmV}{\eta_0}. \tag{4.42}$$

Thus we see that for a cylinder $f \sim V\log V$ at large K, and to make the transition to small Knudsen numbers, one must sum a series in powers of $(\log V)$ which leads to $f \sim$ $\sim V/\log V$ for small K and V.

5. Conclusion

We have discussed the force on a sphere and on a cylinder placed in a gas stream. We considered the case where the Mach number is small and studied the force as a function of the Knudsen number. We generalized the Chapman-Enskog normal solution to the case where a macroscopic object is present, and we also solved the Boltzmann equation by iteration. The latter method was shown to be capable of giving the force on the object at both high and low Knudsen numbers. We may therefore conclude that the

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force on the object is due to the same dynamical events in both limits. However, in the rarefied gas regime the dominant term is the free molecular flow term, while in the hydrodynamic regime the force on the object depends on the fact that the particles make many collisions with the object, and between these collisions they make long excursions into the fluid, traveling many mean free paths.

It is interesting to notice that qualitatively, the force on a sphere is of the form

$$\mathbf{F} \sim \frac{\alpha(\mathrm{K}) R^2 \mathbf{V}}{1+\beta(\mathrm{K}) R/l},$$

where $\alpha(\mathrm{K})$, $\beta(\mathrm{K})$ are functions of the Knudsen number but are of order unity for all values of K. The free molecular flow results and Stokes' Law can be obtained from this formula in the case when $R/l \to 0$ and ∞, respectively. To compute $\mathbf{F}$ in the intermediate region one would need to determine $\alpha(\mathrm{K})$, $\beta(\mathrm{K})$ for all K and these quantities are in turn determined by the structure of the boundary layers around the sphere. To determine α, β we would need a complete solution of the Boltzmann equation, which is very difficult to find, in general. However, Cercignani and co-workers have developed a variational method which appears to give very good approximations to α and β [5].

In the case of the force on a cylinder we have seen that the linear relation $\mathbf{F} = \zeta \mathbf{V}$ breaks down at both small and large Knudsen numbers. This breakdown of the linear laws of hydrodynamics seems to be a general feature of fluid systems with basically two-dimensional geometry. It would be very interesting to know if the divergence difficulties which appear in the theory of transport processes in two-dimensional fluids [20] could be solved as well by the introduction of non-linear relations between the hydrodynamic fluxes and the related driving forces.

Acknowledgments

We would like to thank Drs. E.G.D. COHEN, I. de SCHEPPER, M. DORFMAN, J. V. SENGERS, Y. H. KAN and Dr. D. LIEBERWORTH for many stimulating conversations. This research was supported by the National Science Foundation, Grant No. GP38965, and by the Independent Research Program of the Naval Surface Weapons Center.

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17. That is, the number of conditions imposed on the hydrodynamic variables exceeds the number of parameters that have to be specified.
18. We also assume that Ψ vanishes for $r < R$, so that $W(r)\Psi = \Psi$.
19. In constructing the $\overline{T}$ operator one must be careful to avoid expressions like $\overline{T}PW(r)h(\mathbf{r}, \mathbf{v})$, which are not well defined since products of $\delta(r-R)$ appear. As a result, the correct expression for $\overline{T}$ is slightly different from that given here. For details see Ref. 12.
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