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ON THE CONICS WHICH TOUCH FOUR GIVEN LINES.

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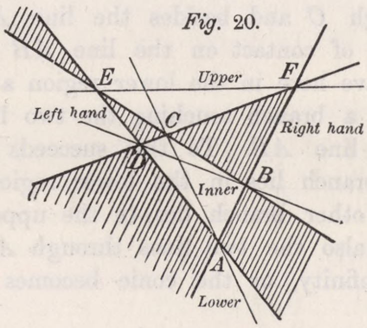
THERE are considerable practical difficulties in drawing a figure of the system of conics which touch four given lines, but a notion of the figure of the system may be obtained as follows:

Figure 20 may be taken to represent any quadrilateral whatever, having all its sides real; and if we attend only to the unshaded spaces, it will be seen that there are five regions which are called the inner, upper, lower, right-hand, and left-hand regions respectively. The inner diagonals are AC the vertical diagonal and BD the horizontal diagonal; the former of these traverses the inner, upper, and lower regions;

Fig. 20.



the latter the inner, right-hand, and left-hand regions; the outer diagonal is EF which traverses the upper region and the right-hand and left-hand regions. The inner or vertical and horizontal diagonals meet in the inner region, the vertical and outer diagonals meet in the upper region, the horizontal and outer diagonals meet in the left-hand region. No conic touching the four lines lies wholly or in part in the shaded regions, and every conic touching the four lines lies wholly in the inner region or wholly in the upper region, or partly in the upper and partly in the lower region, or partly in the right-hand and partly in the left-hand region. It will be convenient to consider, 1°. the conics which lie in the inner region; 2°. the conics which lie in the upper and lower regions, or in the upper region only; 3°. the conics which lie in the right-hand and left-hand regions.

1°. The conics in the inner region are obviously ellipses; an extreme term is the finite right line *BD*, considered as an indefinitely thin ellipse; this gradually broadens out and there is (as a mean term) an ellipse which touches the four lines in the points in which they are intersected two and two by the lines joining the points *E*, *F* with the point of intersection of the diagonals *AC* and *BD*, the ellipse then narrows in the transverse direction and at length reduces itself to the finite line *AC* considered as an indefinitely thin ellipse.

2°. Considering first the conics which lie in the upper and lower regions, these are of course hyperbolas, an extreme term is the infinite portions $A\infty$ and $C\infty$ of the line *AC*, considered as an indefinitely thin hyperbola: we have then hyperbolas such that for each of them, one branch lies in the lower region and touches the two lines through *A*, while the other branch lies in the upper region and touches the two lines through *C*; the points of contact of the lower branch with the lines through *A* gradually recede from *A*, but the point of contact on the line *AD*, which adjoins the left-hand region, recedes with the greater rapidity, and it at last becomes infinite while the point of contact with the line *AB* which adjoins the right-hand region remains finite: we have thus a hyperbola having the line *AD* for its asymptote; the point at infinity of this line belongs of course indifferently to the upper and lower regions; and we may therefore consider one branch as lying in the lower region and touching *AB* and (at infinity) *AD*; the other branch as lying in the upper region and touching the two lines through *C*, and besides (at infinity) the line *AD*. We have next a series of hyperbolas such that for each of them, one branch lies in the lower region and touches only the line *AB*, the other branch lies in the upper region and touches the two lines through *C* and besides the line *AD*. We arrive again at a limiting case when the point of contact on the line *AB* passes off to infinity, or *AB* becomes an asymptote; we have here in the lower region a branch touching (at infinity) *AB* and in the upper region a branch touching the two lines through *C*, the line *AD*, and besides (at infinity) the line *AB*. To this succeeds a series of hyperbolas such that for each of them, one branch lies in the lower region but does not touch either of the lines through *A*, the other branch lies in the upper region and touches as well the two lines through *C* as also the two lines through *A*. Finally the branch in the lower region passes off to infinity, or the conic becomes a parabola lying wholly in the upper region.

We have thus arrived at the conics which lie wholly in the upper region, the extreme term being a parabola: this passes into an ellipse touching of course the four lines, and gradually reducing itself to the finite line *EF* considered as an indefinitely thin ellipse.

3°. We come now to the conics which lie in the right-hand and left-hand regions; an extreme term is the conic composed of the infinite portions $E\infty$ and $F\infty$ of the line EF, considered as an indefinitely thin hyperbola; we have then a series of hyperbolas such that for each of them the branch in the right-hand region touches the two lines through E, and the branch in the left-hand region the two lines through F; we then arrive at a limiting case where the branch in the right-hand region touches the line EC the upper boundary of the right-hand region at ∞, or where EC is an asymptote; we may say that the branch in the right-hand region touches the line EB and (at infinity) the line EC, while the branch in the left-hand region touches not only the two lines through F but also (at infinity) the line EC; we have next a series of hyperbolas such that for each of them the branch in the right-hand region touches only the line EB while the branch in the left-hand region touches as well the two lines through F as the line EC; we have then again a limiting case, viz. the branch in the right-hand region touches the line BC at infinity, or BC is an asymptote; we may here say that the branch in the right-hand region touches the line BE and (at infinity) the line BC, while the branch in the left-hand region touches the lines through D and (at infinity) the line BC; and to this succeeds a series of hyperbolas such that for each of them the branch in the right-hand region touches the two lines through B while the branch in the left-hand region touches the two lines through D; the hyperbola finally becomes the infinite portions $B\infty$ and $D\infty$ of the line BD, considered as an indefinitely thin hyperbola.

It is to be noticed that the entire system commences with the finite line BD considered as an indefinitely thin ellipse and passes on continuously to the infinite portions $B\infty$ and $D\infty$ of the same line, considered as an indefinitely thin hyperbola: it is therefore a cyclical series, and any term whatever might have been taken for the commencement of the series.