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Approximate path independent integrals in the plane problems of cracks and associated antiplane shear problems

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THE paper discusses the path independence property of generalized integrals in the plane problems of cracks J_k and I_k, the problem of a branched crack and approximation of path integrals utilizing the bounds of the potential energy. Upper and lower bounds for the potential energy are furnished by the two antiplane shear problems associated with the plane strain problem, which are solved experimentally by standard methods of potential flow fields.

W pracy przedyskutowano szczegółowo własność niezależności od drogi całek uogólnionych J_k i I_k występujących w płaskich zagadnieniach szczelin, problem rozgałęzionej szczeliny oraz aproksymację całek wykorzystując oszacowania energii potencjalnej. Górnego i dolnego oszacowania energii potencjalnej dostarczają dwa zagadnienia antypłaskiego ścinania stowarzyszone z zagadnieniem płaskiego stanu odkształcenia, które zostały rozwiązane doświadczalnie metodami standardowym dla przepływów w polach potencjalnych.

В работе обсуждены подробно свойство независимости от пути обобщенных интегралов J_k и I_k, выступающих в плоских задачах трещин, проблема разветвленной трещины, а также аппроксимация интегралов, используя оценку потенциальной энергии. Верхней и нижней оценки потенциальной энергии доставляют две задачи антиплоского сдвига ассоциированные с задачей плоского деформационного состояния, которые были решены экспериментально стандартными методами для течения в потенциальных полях.

1. Introduction

THE J-integral is extensively used in fracture mechanics as a new method of getting stress-intensity factors and energy release rate [1, 2]. This paper discusses the path independence property of certain generalized integrals J_k introduced by BUDIANSKY and RICE [3], as well as the significance of the J-integral use in the case of branched cracks.

Path independence property is not satisfied when approximate fields are introduced, as shown in papers [4, 5] dealing with the dual formulation of the J theory. The dual integral I is first introduced in [4]. Its generalization I_k is given in [6] and also in paper [7] by CARLSSON, but not in connection with the problem of duality between two independent kinematic and static fields. The use of such fields as an approximation yields upper and lower bounds for the potential energy. In this paper another set of bounds is given for a plane strain case, based on the two associated antiplane shear problems.

Analogous methods for getting the stress intensity factor K_{III} are also given.

2. Path-integrals J_k and I_k

The integrals J_k and I_k ($k = 1, 2$) are defined as follows:

(2.1)
$$J_k^C = \int_C \{W(\varepsilon)n_k - \sigma_{ij}n_j u_{i,k}\}ds,$$
$$I_k^C = -\int_C \{U(\sigma)n_k - u_i n_j \sigma_{ij,k}\}ds,$$

where C is an *open* path joining two arbitrary points on opposite sides of the crack's surface while going around the tip; $\mathbf{n}$ is a unit outward normal to C; the comma means partial derivative. In the brackets we have the elastic strain energy density $W(\varepsilon)$ and its Legendre transform $U(\sigma)$ which give stress-strain laws in the alternative forms:

(2.2)
$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}, \quad \varepsilon_{ij} = \frac{\partial U}{\partial \sigma_{ij}}.$$

In this paragraph, compatibility and equilibrium are supposed to be verified:

(2.3)
$$\varepsilon_{ij} = (u_{i,j}+u_{j,i})/2, \quad \sigma_{ij,j} = 0.$$

From (2.2), (2.3) it follows that

(2.4)
$$\oint \{W(\varepsilon)n_k - \sigma_{ij}n_j u_{i,k}\}ds = 0,$$
$$\oint \{U(\sigma)n_k - u_i n_j \sigma_{ij,k}\}ds = 0,$$

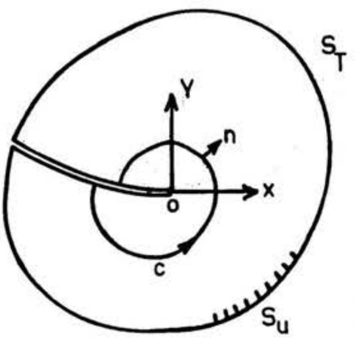

FIG. 1. Path integration for a curved crack in an elastic body.

for all *closed* paths not including holes or singularities. Hence, it follows from the conservation's laws (2.4), that if the integrands vanish along the crack's surface, then J_k and I_k are independent of the path's end points (Fig. 1). This is a strong condition which restricts considerably the applicability of the path independence theorem to particular cases. Consider a straight crack along the x axis, then J_x and I_x are known to be path independent. If the geometry and loading are such that $W^+ = W^-$ on the crack's surface, then, J_Y is path independent, but the two end points must be opposite to each other.

For any curved crack which is tangent to $0x$ like that shown in Fig. 1, there is no path independence. Consider now two particular path-integrals: J_K^S along the outer contour S and J_k^0 along a zero circuit C_0 around the tip. Then,

(2.5)
$$J_k^S \neq J_k^0$$

and similar inequality for the dual integral I_k. This may be interpreted by energy considerations. It is worthwhile dwelling on the physical significance of these path integrals.

3. Energy considerations

The interpretation of J_k and I_k as "generalized crack extension forces" is based on their equivalence with the potential energy release rate. Let the potential energy of the body be P and its complementary Q:

$$
P(\varepsilon) = \int_{\Omega} W(\varepsilon)dA - \int_{S_T} \overline{T}_i u_i ds,
$$

(3.1)

$$
Q(\sigma) = -\int_{\Omega} U(\sigma)dA + \int_{S_u} \overline{u}_i \sigma_{ij} n_j ds,
$$

where $\overline{T}_i$ is the prescribed traction on S_T and $\overline{u}_i$ the prescribed displacement on S_u.

Let the crack be translated with the unit velocity l while the exterior boundary remains fixed and the boundary conditions remain unchanged; the derivative of the potentials with respect to a time like parameter t is given by [1, 3, 4, 14]

$$
\frac{dP}{dt} = -l_k J_K^S, \qquad \frac{dQ}{dt} = -l_k I_k^S. \tag{3.2}
$$

Thus, the integrals J_k^S and I_k^S with signs reversed are considered as generalized crack extension forces. They are associated with a translation of the whole crack, which generally is not the same as that growing at the tip by the same amount, like in the case of straight crack moving in its plane. Generally, the virtual velocity field defined on the crack does not correspond to any physical process. Clearly, any velocity field for a virtual growth of a crack of any form must have vanishing normal component on the main part of the crack but not near the tip. If the crack grows smoothly, i.e., tangently at the tip, the energy release rate is given by Irwin formulas or by the integrals J_x^0 or I_x^0 associated with the zero circuit:

$$
J_x^0 = I_x^0 = \frac{\varkappa+1}{8\mu}(K_I^2 + K_{II}^2), \tag{3.3}
$$

$\varkappa = 3-4\nu$ in plane strain, $\varkappa = (3-\nu)/(1+\nu)$ in plane stress case, K_I and K_{II} are the stress-intensity factors.

For a straight crack, a virtual translation in the y-direction normal to the crack leads to

$$
J_Y^0 = I_Y^0 = -\frac{\varkappa+1}{4\mu} K_I K_{II} \tag{3.4}
$$

(cf. BERGEZ [8], CARLSSON [7]).

While (3.3) has been interpreted physically as the energy release rate for extension in the x-direction, Eq. (3.4) has not.

The concept of a virtual motion given by a variable velocity field $v_i(s)$ used by BUDIANSKY and RICE for a cavity [3] can be extended for a crack as follows:

$$
\frac{dP}{dt} = -\int_{C} (Wn_k - \sigma_{ij} n_j u_{i,k}) v_k ds \tag{3.5}
$$

with the path C in Fig. 2 consisting of the cracks' faces C^+ C^- and the zero circuit C_0 around the tip. The particular integrals J_k^S and J_k^0 are then related to Eq. (3.5) through the velocity fields: For $v_k(s) = l_k$ Eq. (3.5) gives the energy release rate $-l_k J_k^C$; the path C can be replaced by S. For $v_k(s) = \tau_k(s)$, where $\boldsymbol{\tau}$ is the unit vector tangent to the crack, Eq. (3.5) gives the energy release rate $-\tau_k(0) J_k^0$.

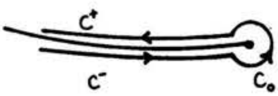

FIG. 2. Path integration along the crack.

4. The problem of the branched crack

The problem of crack growth due to an abrupt change of direction (Fig. 3), has been studied by many authors [9, 10, 12]. However, the applicability of path integrals for this problem is not clear. This is a typical case of path dependence of the integral. Consider two paths C_1, C_2 for the branched crack and the cases where the length of the branch shrinks to zero as well as the paths radius, $l \to 0$, $C_i \to 0$. Those limits must certainly be taken in suitable order if it is to influence the final result. The results obtained may vary according to path dependency and difference in the limiting singular fields by a change of the polar coordinate. When $l = 0$, the zero path C_1 goes around the tip A of the original crack OA, so that the integral $J_k^{(1)}$ is computed from the initial field of the main crack or from the stress-intensity factors K_I, K_{II}. Now, the second integral should be defined as

$$J_k^{(2)} = \lim_{l \to 0} \lim_{C_2 \to 0} J_k^{C_2}. \tag{4.1}$$

New singular fields at the tip B are needed to compute (4.1). For example, consider the branch crack (Fig. 3) under remote uniform traction T at the angle β to the main crack. Exact values of K_I, K_{II} at the tip A of the initial crack OA ($l = 0$) yields cf. [7]:

$$J_\alpha^{(1)} = \frac{\varkappa + l}{8\mu} \pi a T^2 \sin^2\beta (\cos\alpha - \sin 2\beta \sin\alpha). \tag{4.2}$$

This integral should be interpreted as a generalized force associated with the translation of the crack in the direction making an angle α with Ox. Attemps to solve analytically the problem of the branched crack are found in [9, 10]. For our purpose we take the new stress-intensity factors K_I^*, K_{II}^* given by HUSSAIN, PU, UNDERWOOD [9]. These factors at the tip B are associated with the new polar coordinates ϱ, ϕ.

The limiting values for $l = 0$ are:

$$\begin{aligned} K_1^* &= T(\pi a)^{1/2} \frac{4}{3+\cos^2\alpha} \left(\frac{1-m}{1+m}\right)^{m/2} \left(\cos\alpha \sin^2\beta + \frac{3}{2} \sin\alpha \cos\beta \sin\beta\right), \\ K_{II}^* &= T(\pi a)^{1/2} \frac{4}{3+\cos^2\alpha} \left(\frac{1-m}{1+m}\right)^{m/2} \left(\cos\alpha \sin\beta \cos\beta - \frac{1}{2} \sin\alpha \sin^2\beta\right). \end{aligned} \tag{4.3}$$

The $J_\alpha^{(2)}$ integral is given by ($m = \alpha/\pi$)

$$J_i^{(2)} = J_x^{(2)} \cos\alpha + J_y^{(2)} \sin\alpha = \frac{\varkappa+1}{32\mu} \pi a T^2 \left(\frac{1-m}{1+m}\right)^m \left(\frac{4}{3+\cos^2\alpha}\right)^2 \times \tag{4.4}$$

$$\times \left((1+3\cos^2\alpha)\sin^4\beta + 8\sin\alpha\cos\alpha\sin^3\beta\cos\beta + (9-5\cos^2\alpha)\sin^2\beta\cos^2\beta\right)$$

which can be put in the familiar form

$$J_{\alpha}^{(2)} = \frac{\varkappa+1}{8\mu}(K_I^{*2}+K_{II}^{*2}). \tag{4.5}$$

Therefore, the integral (4.4) is interpreted as the energy release rate in the extension of the crack in the α-direction which is different from the original x-direction. Equations (4.4) and (4.2) clearly show the path dependence of the J-integral.

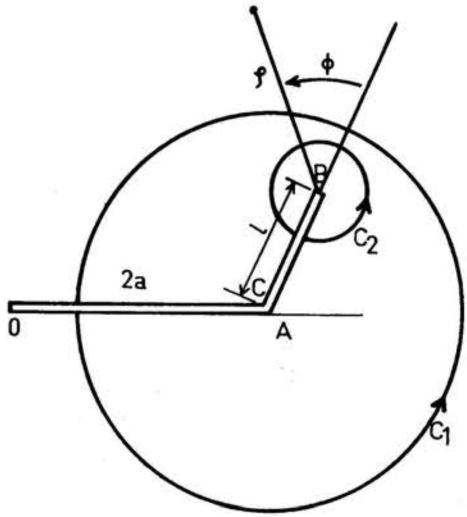

FIG. 3. Branched crack.

Without referring to the path-integral, the energy release rate can be defined directly as the work required to close the branch ABC

$$G = \lim_{dl\to 0}\frac{1}{2dl}\int_{ABC} u_i(1+dl)\,T_i(l)\,ds \tag{4.6}$$

which is equivalent to

$$G = \frac{1}{2}\int\left(T_i\frac{\partial u_i}{\partial l}-u_i\frac{\partial T_i}{\partial l}\right)ds, \tag{4.7}$$

where the integral is taken over any closed curve enclosing the crack. Formulas (4.6) and (4.7) apply to linear elastic materials only. Equation (4.5) shows the equivalence between the $J_\alpha^{(2)}$ integral and the energy release rate G.

DUDUKALENKO and ROMALIS [10] have computed G for $l = 0$. Their result is reproduced here:

$$G = \frac{\pi(\varkappa+1)aT^2}{8\mu}\left(\frac{1-m}{1+m}\right)^m\Bigg((\sin^2\beta+\frac{1}{4\pi}\sin\alpha\cos(2\beta-\alpha)\left(\frac{2m}{1-m^2}-\log\frac{1-m}{1+m}\right)$$
$$+\frac{1}{32\pi}\sin 2\alpha\left(\frac{8m}{1-m^2}+3\log\frac{1-m}{1+m}\right)+\frac{1}{16\pi^2}\sin^2\alpha\Bigg(\frac{3\pi^2}{2}-\frac{4m}{1-m^2}\log\frac{1-m}{1+m}$$
$$-\frac{m}{(1+m)^3}\log\frac{1-m}{1+m}+32+\frac{1}{2(1-m^2)}\left(\log\frac{1-m}{1+m}\right)^2+\frac{2}{(1+m)^3}-\log\frac{1-m}{1+m}\Bigg)\Bigg). \tag{4.8}$$

It can be seen that results (4.4) and (4.8) are different. Without explaining this disagreement, it turns out that the growth directions predicted according to the greatest energy released are very close to each other (Fig. 4).

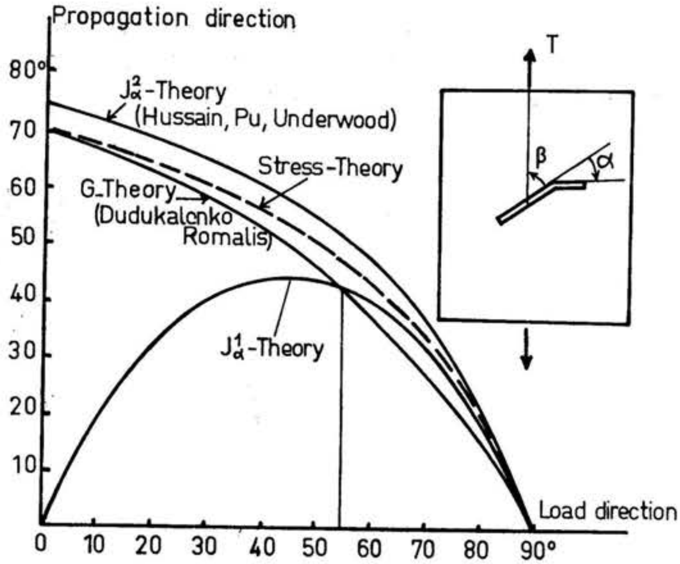

FIG. 4. Crack growth direction versus loading angle according to the greatest energy released.

For comparison curves are plotted in the same figure of values obtained from the $J_\alpha^{(1)}$ theory:

$$\sin\alpha + \sin 2\beta \cos\alpha = 0 \tag{4.9}$$

and from the maximum stress-theory, cf. [11]

$$\sin\alpha - (3\cos\alpha - 1)\frac{\cos\beta}{\sin\beta} = 0. \tag{4.10}$$

For small β, Eq. (4.9) gives for the crack growth direction a value not far from zero, while other curves give $-\alpha$ not far from 70°. CHATTERJEE [12] gives a numerical solution for the same problem but for $l/2a$ higher than 10^{-3}. Inserting the K_I^* and K_{II}^* values given by him in Eq. (4.5), we find values different from (4.4) and (4.8) (Table 1).

Table 1.

$4\mu G/(\varkappa+1)aT^2 \quad (l = \pi/2)$

	Ref. 9	Ref. 10	Ref. 12
$\alpha = \pi/2$	0.23	1.13	0.34
$\alpha = 0$	$\pi/2$	$\pi/2$	

The lack of accuracy of the numerical solution and the fact that the branch length is not zero in [12] undoubtedly make the comparison difficult.

5. Dual variational method

Equations (2.1), (3.2) show the symmetry between the theories of the J and I integrals and the equality between integrals which follows from

$$(5.1) \qquad P(\varepsilon) = Q(\sigma)$$

whenever stress and strain fields are actual solutions of the boundary-value problem [5]. Paper [5] discusses approximate solutions in relation with the dual variational method. Accordingly, approximate fields are components of two independent ones: kinematic ε^* and static σ^{**}. The results of [5] are summed up as follows: for approximations, the integrals in $(2.1)_1$ and $(2.1)_2$ are path dependent and $(3.2)_1$ and $(3.2)_2$ themselves are not valid. What remains is the so-called bound-theorem

$$(5.2) \qquad -P(\varepsilon^*) < -P(\varepsilon) = -Q(\sigma) < -Q(\sigma^{**})$$

which holds for any material, the energy density of which is a convex function of the strain (for W). Starting with approximate potentials $P^* = P(\varepsilon^*)$ and $Q^{**} = Q(\sigma^{**})$ approximate values of integrals J^S amd I^S are defined by

$$(5.3) \qquad J^* = -\frac{\partial P^*}{\partial l}, \quad I^{**} = -\frac{\partial Q^{**}}{\partial l}.$$

We consider here only the case of a straight crack growing in its plane. As pointed out in [5], the gradients (5.3) *are not* respectively lower and upper bounds for the integrals J or I, neither are they values of the integrals as given by direct calculations according to their definitions (2.1) from the approximate fields. This limitation should be observed seriously together with the fact that no information is yet available besides that given by (5.3). Equation $(5.3)_1$ is usual in the finite element method which deals with the kinematic fields. It is found that the calculated value of the J integral by Eq. $(5.3)_1$ is generally underestimated just like the value of $-P$. If examples of bounding the J-integral by means of Eqs. (5.3) can be given, cf. [5], unfortunately there is no analytical criterion which would give inequalities similar to (5.2) for the gradients of the potentials. Of course, $J^* < I^{**}$ is necessary but not sufficient.

In the following paragraph bounds for the potential P will be presented by associated antiplane shear problems.

6. Associated antiplane problems

Let a linear elastic body be subjected to tractions T_x, T_y with no body forces. Equilibrium in the x-direction gives

$$\sigma_{xx,x} + \sigma_{xy,y} = 0 \quad \text{in } \Omega,$$
$$\sigma_{xx} n_x + \sigma_{xy} n_y = T_x \quad \text{on } S.$$

To get a first static field $\sigma^{**}(T_x)$ for the antiplane loading by the longitudinal shear $\tau = T_x$, the following field is used:

$$\sigma_{zx}^{**(1)} = \sigma_{xx}, \quad \sigma_{zy}^{**(1)} = \sigma_{xy}.$$

Similarly, a second static field is derived from the remaining equation of equilibrium and boundary condition

$$\sigma_{zx}^{**(2)} = \sigma_{yx}, \quad \sigma_{zy}^{**(2)} = \sigma_{yy}.$$

The corresponding stress potentials are

$$\text{(6.1)} \qquad \begin{aligned} Q_1^{**} &= -\frac{1}{2\mu}\int_\Omega (\sigma_{xx}^2+\sigma_{xy}^2)dA, \\ Q_2^{**} &= -\frac{1}{2\mu}\int_\Omega (\sigma_{yx}^2+\sigma_{yy}^2)dA. \end{aligned}$$

We return to the plane strain problem for which we want to bound the potential $Q(\sigma)$ as follows

$$Q(\sigma) < \frac{1-2\nu}{2}\int_\Omega -\frac{1}{2\mu}(\sigma_{xx}^2+2\sigma_{xy}^2+\sigma_{yy}^2)dA.$$

According to (6.1) the potential Q is bounded by

$$\text{(6.2)} \qquad Q(\sigma) < \frac{1-2\nu}{2}(Q_1^{**}+Q_2^{**}).$$

By (5.2) we can replace $Q_1^{**}Q_2^{**}$ by their exact values without changing inequality (6.2). Hence, the exact stress potentials of the associated antiplane problems give upper bounds to $Q(\sigma)$.

To get a lower bound for Q or P, the potential $P(\varepsilon)$ is written in the form

$$P(\varepsilon) = \alpha^2\int_\Omega 2W(\varepsilon)dA - \beta^2\int_S T_i u_i ds$$

with $\beta^2-\alpha^2 = 1/2$ so that the value of $P(\varepsilon)$ is unchanged. The first integral is bounded according to Korn's inequality

$$\int_\Omega 2W(\varepsilon)dA > k_0\int u_{i,k}u_{i,k}dA$$

where k_0 is a constant. A possible one is $k_0 = \mu$ though it is not the best one. Choosing $\mu\beta^2 = 2k_0\alpha^2$

$$\beta^2\left\{\frac{\mu}{2}\int_\Omega u_{i,k}u_{i,k}dA - \int_S T_i u_i ds\right\} < P(\varepsilon).$$

From straightforward calculation it follows that the bracket is the sum of the strain potential energies P_1^* and P_2^* of the associated antiplane problems

$$\text{(6.3)} \qquad \beta^2(P_1^*+P_2^*) < P(\varepsilon).$$

The kinematic fields of the associated problems are

$$w^{*(1)} = u_x, \quad w^{*(2)} = u_y.$$

The approximate potentials $P_1^* P_2^*$ can be replaced by their exact values and the inequality (6.3) remains valid. In conclusion, the plane strain potential energy can be bounded on both sides by using the associated antiplane problems. The bounds are not good enough for an estimation of the J integral but they give useful indications as to the potential P. They are easier to calculate in antiplane shear condition than in plane strain condition. Moreover, the associated antiplane problems give roughly the same stress-intensity factors as in plane strain condition for problems which have similar boundary values. This will be illustrated by two examples.

Example 1. An infinite body is subjected to a remote uniform traction T perpendicular to a crack with the length $2a$. The stress intensity factor normalized by $T(\pi a)^{1/2}$ is $K_I = 1$. The antiplane problem associated with this boundary condition gives $K_{III} = 1$.

Example 2. An edge-crack with the length a, the crack being parallel to Ox, the boundary condition being a traction T at infinity in the y-direction. The stress-intensity factor is $K_I = 1.12$ while the corresponding antiplane problem gives $K_{III} = 1$. That is a difference of 12% which is small when compared to the uncertainty of the experimental critical value.

At the moment no analytical treatment is available for finding boundary conditions for plane strain problems the stress-intensity factors of which come within a given range of those of the corresponding antiplane deformation problems.

7. Fluid analogy

For antiplane shear loading, stresses σ_{zx}, σ_{zy} and displacement w can be given by one analytical function $f(Z)$ of the complex variable $Z = x+iy$

$$\sigma_{zx} - i\sigma_{zy} = f'(Z),$$

$$w = \mathrm{Re} f(Z), \quad J_x - iJ_y = i/2\mu \oint (f')^2 dZ.$$

Similar relations are known for two-dimensional irrotational flows of incompressible non-viscous fluid with the velocity $\mathbf{V} = \nabla\phi$

$$V_x - iV_y = f'(Z), \quad \phi = \mathrm{Re} f(Z), \quad F_x - iF_y = i\varrho/2 \oint (f')^2 dZ.$$

In this analogy the complex J-integral appears to be equal to the hydrodynamic force acting on the tip of a thin plate. For boundary conditions, the normal velocity is made equal to the shear. Of course, disturbing effects like viscosity and separation of the boundary layer must be estimated.

Another simple method uses the analogy between the I-integral and the kinetic energy variations of the fluid with respect to the plate length. Another analogy is possible, namely, the flow of an electric current j on a graphite-coated paper having a straight slit (length a) simulating the crack while normal current j_n is maintained on the boundary, cf. [13]

$$j_n = c(\sigma_{zx} n_x + \sigma_{zy} n_y),$$

c is a suitable dimensional constant. The power dissipated Π by Joule's effect is related to the I-integral

$$I = (2\mu R c^2)^{-1} \frac{d\Pi}{da},$$

R is the specific resistivity of the paper. Results of experiments on a finite strip having an edge crack are reported in [13] and agree well with calculations from an infinite strip (for a height greater than twice the width it is better than 1% Fig. 5).

The analogy is even more valuable in the case of many cracks or holes of any shape, for which no analytical approach is possible. Rigid inclusions can also be represented by highly conductive areas ($R = 0$), for example, by using conductive paint coatings.

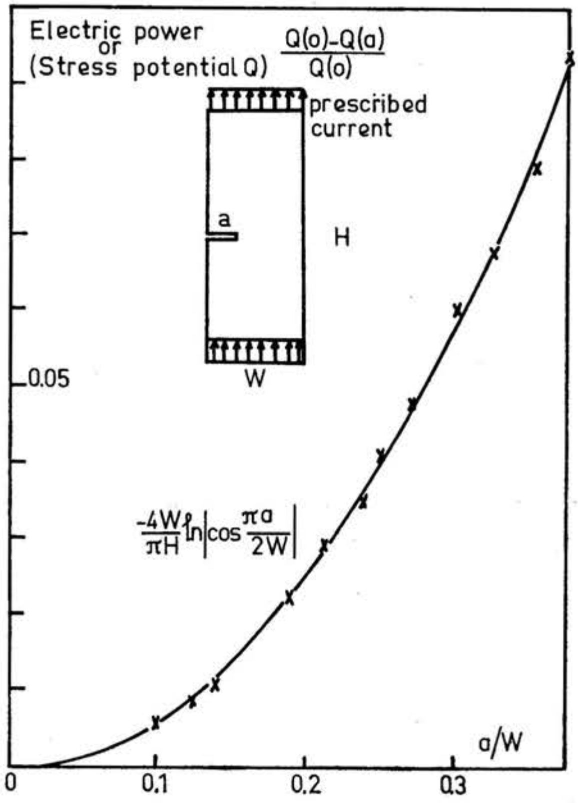

Fig. 5. Edge-crack in a finite strip simulated by a two-dimensional flow of electric current on a graphite coated paper.

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