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On the non-Gausian aspects of turbulence

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THE TURBULENT energy dissipation is controlled by non-linear processes directly related to the skewness of certain velocity derivatives. Experimental studies have shown that non-Gaussian properties are associated with sharp spikes of the derivatives. We have used three dimensional time dependent numerical models of turbulence in physical space, starting from random conditions, to study the non-Gaussian effects. We found that the physical mechanism associated with non-Gaussian effects is analogous to collision between two round jets, in the presence of transversal vorticity.

O turbulentnej dysypacji energii decydują procesy nieliniowe, na które wpływają bezpośrednio niektóre pochodne prędkości. Jak wynika z badań doświadczalnych, niegaussowskie własności są związane z występowaniem ostrych pików pochodnych. W pracy tej badano efekty niegaussowskie przy użyciu numerycznych modeli turbulencji, w przestrzeni fizycznej. Analizowano modele trójwymiarowe, z uwzględnieniem zmiennej czasowej, przyjmując losowe warunki początkowe. Stwierdzono, że fizyczne mechanizmy, odpowiedzialne za efekty niegaussowskie, są podobne do tych, które określają zderzenie między dwoma okrągłymi strugami, przy występowaniu poprzecznej wirowości.

О турбулентной диссипации энергии решают нелинейные процессы, на которые влияют непосредственно некоторые производные скорости. Как следует из экспериментальных исследований негауссовские свойства связаны с выступанием острых пиков производных. В данной работе негауссовские эффекты исследованы при использовании численных моделей турбулентности в физическом пространстве. Анализированы трехмерные модели с учетом временной переменной, принимая случайные начальные условия. Констатировано, что физические механизмы, ответственные за негауссовские эффекты, аналогичны тем механизмам, которые определяют столкновения между двумя круглыми струями, при выступании поперечной завихренности.

Introduction

MUCH of what we know about turbulence is derived from the analysis of the signal produced by a single hot wire showing the velocity fluctuation as function of the time, as illustrated in Fig. 1a. By Taylor's [Ref. 1] hypothesis, this is like knowing the fluctuation $u(x, y, z, t)$ as function of x since the fluid passes by the hot wire with a much larger velocity.

The probability distribution of such signals is nearly Gaussian. The time derivative of the signal, (see Fig. 1 b.) is not Gaussian, as first reported by TOWNSEND [Ref. 2]. It has a clear preference for positive peaks. These positive spikes correspond to jumps from a velocity below average to one above average, as shown at times t_1, t_3 and t_4. The reverse jumps are generally weaker and less frequent, as shown for time t_2. Because of these spikes, the probability distribution of the derivative has an exponential tail. While the mean value of the derivative is null, its mean cube is not. This is the well-known skewness factor, proportional to Kolmogoroff's constant, or to the rate of energy trans-

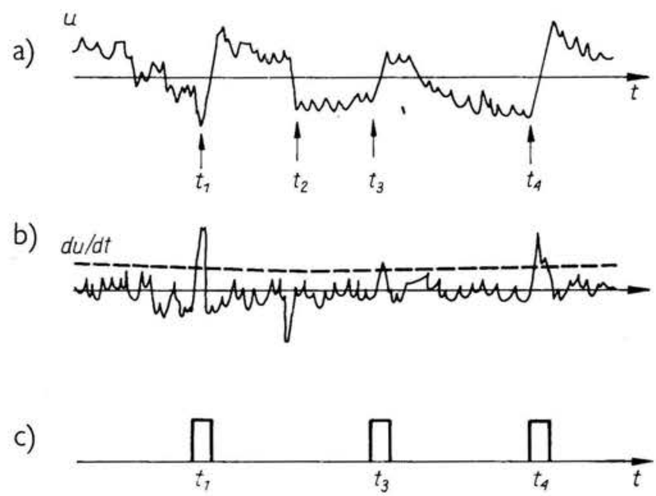

FIG. 1. a) Velocity fluctuation u, as function of time t. The mean flow rapidly transports the fluid past the anemometer; b) time derivative of the above signal; c) electronic pulses produced after each crossing of an arbitrary constant level.

fer to the small eddies. If we disregard the spikes, it seems that the derivative has nearly a Gaussian distribution. The present study of the non-Gaussian aspects of turbulence begins with the properties of these spikes. Numerical experiments were performed and the details of successive steps are reported in Ref. [3, 4 and 5].

1. Properties of the spikes

A simple electronic circuit is used to produce standard pulses after each excursion of the signal du/dt above an arbitrary level. The pulses obtained at t_1, t_3 and t_4 are shown in Fig. 1c. Their duration is somewhat longer than that of the spikes. In a strongly

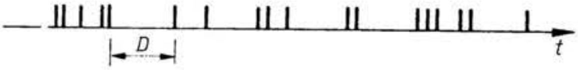

FIG. 2. The pulses occur in clusters.

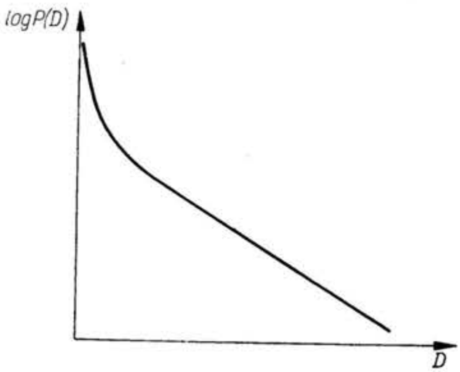

FIG. 3. Probability distribution of the time interval between successive pulses, in semi-long scales. About one half of the pulses are separated by short intervals.

turbulent flow obtained by mixing 125 air jets, it was noticed that the pulses tend to occur in groups separated by quiet intervals (see Fig. 2). The probability distribution of the time interval D between successive pulses was examined. If the pulses occurred at random times, as in a typical radioactivity experiment, the probability distribution would be exponential. The present results can be approximated as a two-component distribution (see Fig. 3). This means that about half of the pulses occur in clusters, separated by randomly distributed intervals. So far no evidence has been found within a group, suggesting that the pulses are associated with any simple geometry such as a plane vortex sheet or a straight vortex tube. The scale of the groups is probably the same in all directions, so that most pulses occur within a small fraction of the fluid volume evaluated at 15%.

2. Phase relations

The signal sketched in Fig. 1 can be described by a Fourier series:

$$u(t) = \sum_{i=1}^{\infty} A_i \sin(\omega_i t + \varphi_i). \tag{2.1}$$

The frequencies uniformely cover the range from 0 to some upper limit. The amplitudes A_i are related to the energy spectrum of the turbulence and become negligible beyond the so-called viscous cut-off.

Let us now define the structure functions $C_N(\tau)$ as follows:

$$C_N(\tau) = \overline{[u(t+\tau)-u(t)]^N}. \tag{2.2}$$

The bar indicates the time average. The variable τ corresponds, by Taylor's hypothesis, to the distance between two points along a stream line of the mean flow. Functions for $N = 2$ to 8 were studied. (For parallel mean flow, $C_1(\tau) = 0$).

The function C_2 depends only upon the amplitudes and cannot tell us anything about the phases. It is directly related to the energy spectrum of the turbulence. Hence, it is not of primary relevance to the present study.

If the phases φ_i were statistically independent, the structure functions of odd order would vanish. Thus the skewness would be null and there would be no energy transfer to the small scales of motion.

Furthermore, we would have

$$\begin{aligned} C_4(\tau) &= 3(C_2(\tau))^2, \\ C_6(\tau) &= 15(C_2(\tau))^3, \\ C_8(\tau) &= 105(C_2(\tau))^4. \end{aligned} \tag{2.3}$$

Thus, with random phases the set of amplitudes determines everything. But it is a fact that in turbulent flows C_3, C_5 and C_7 are not small. Assume that the phases are linked in groups of three only, and not in longer chains. This means that for three frequencies such that $\omega_i = \omega_j + \omega_k$ we could have the relation

$$\varphi_i = \varphi_j + \varphi_k. \tag{2.4}$$

For signals presenting a broad band spectrum, this leads to the approximate relation

$$C_5(\tau) = 10 C_3(\tau) C_2(\tau). \tag{2.5}$$

Factor 10 is related to the fact that five musicians can form one trio and one duet in ten different ways. Similarly,

$$C_7(\tau) = 105\, C_3(\tau)\big(C_2(\tau)\big)^2. \tag{2.6}$$

Measurements of the ratio C_5/C_3C_2, (see Ref. [3]. and [6 through 10]) show values in the range of 9 to 11, provided that the separation τ is large enough. A typical result is sketched in Fig. 4a. The function C_7 shows the same general behavior (see Fig. 4b).

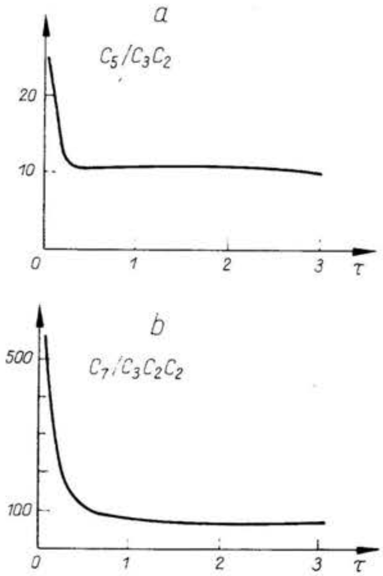

FIG. 4. Measurements of structure functions, in turbulence.
a) Structure function of order 5, made non-dimensional by reference to C_3 and C_2. The value 10 is typical for signals with phases relations limited to triplets. b) Structure function of order 7, non-dimensional. The value 105 is expected for signals with phases relations limited to triplets.

Thus, the assumption of phases relations by triplets only is supported by the experiments with the exception of the short scale phenomena. This means that the spikes are related to a higher order of phase relations and that they cannot be explained by any theory limited to triplet interactions between the Fourier components.

Kraichnan proposed a theory based on the assumption that the interaction between triplets of Fourier components can be described by average response functions (Ref. [11, 12]). The fact that for small values of τ, C_5 rises above the value expected for weak phase relations seems incompatible with Kraichnan's theory. However, to this day a better theory of turbulence does not exist.

If artificial random functions with jumps are generated such as shown in Fig. 1 and if the magnitude and rate of occurrence of the jumps are adjusted so that C_3 resembles the turbulent data, it can be found that C_5 and C_7 behave like their turbulent counterparts. Furthermore, once the odd-order structure functions imitate turbulence, those of even order do likewise. Thus, the jumps of u are essential and seem to account for the main properties of the structure functions.

3. Numerical model

Integration of the Euler equations of motion for incompressible flow inside a box of $55 \times 55 \times 55$ grid points was programmed on a computer. Periodic boundary conditions were used (whatever leaves on the east side reenters on the west). The computer handled 3 velocity components at each grid point, nearly half a million floating point numbers (32 bits each). The R. M. S. velocity fluctuations were set at unity. Integral scales (size of energy carrying eddies) in the range of 7 to 10, and times step of 0.02 were used. Our algorithm was less rigorous than that used by ORSZAG and PATTERSON (Ref. [13]) but more compatible with resources in machine time and storage facilities. Initial conditions were generated such that the phases were statistically independent, and therefore the functions C_3, C_5 and C_7 were initially null. The evolution of C_3, for points two grid meshes apart, is shown in Fig. 5. The independent variable is the time T, corresponding to the age of the turbulence and not to a scanning of the x-axis by a fast moving anemometer as in the case of Fig. 1.

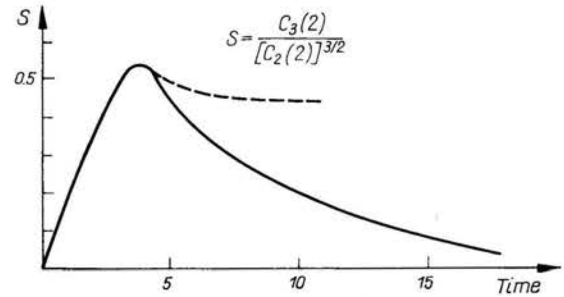

FIG. 5. Skewness factor for a numerical model of turbulence. The structure functions C_3 and C_2 are determined for points two meshes apart. The time corresponds to real time in a turbulent flow without mean motion.

Solid line: inviscid model; dotted line: results with a sufficiently large viscosi

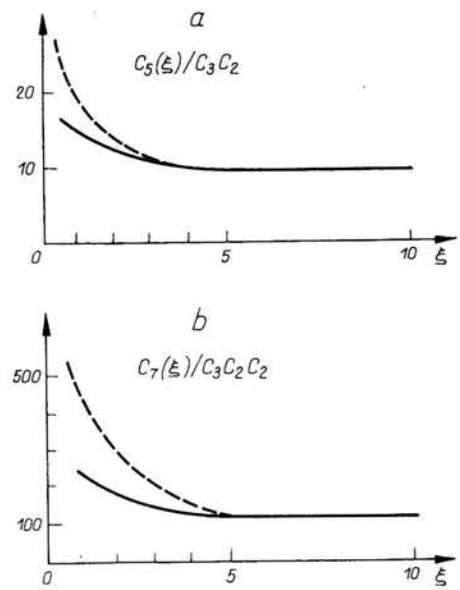

FIG. 6. Structure functions with spatial separation.

Solid line: numerical model, ξ measured in units of one mesh. Dotted line: turbulence, with $\xi = 10\tau$, in order to match integral scales.

Without viscosity the skewness raises to a maximum corresponding to realistic values. Later it decays because the energy, which cannot escape from the box, has been transferred to the finest possible scale of motion. Thus, the grid mesh becomes analogous not only to the Kolmogoroff length but also to a mean free path. The energy simply cannot spread further. The system approaches equipartition of its energy.

The addition of viscous forces to the model, with a sufficiently large viscosity, removes this fine scale random component, and the skewness tends to settle to a constant level, while the energy decays slowly.

The nondimensional forms for structure functions of order 5 and 7 obtained from the numerical models are shown in Fig. 6a and 6b. While they do not raise as high as the turbulent ones, the trend is unmistakable. The dotted lines correspond to turbulent experiments, with matching integral scales. However, the Reynolds number of the numerical model, based on a mesh-size concept, would be much lower than that of the turbulence.

These results suggest that the numerical model has captured not only the simple relations between triplets of Fourier components, but some of the more complex ones.

4. Vorticity and deformation

The numerical model has many obvious shortcomings: the mesh size is too small, the integration procedure is imperfect. But it has one great advantage: it provides us with half a million hot wire anemometers, which do not perturb the flow. At every point we have determined not only the vorticity defined as

$$\Omega_i = \Sigma_{ijk}\, \partial u_j/\partial x_k \tag{4.1}$$

but also the six components of the rate of deformation tensor defined as

$$S_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}. \tag{4.2}$$

At every point, a rotation of the coordinates can be found such that S_{ij} reduces to the eigenvalues

$$S_{ij} = \begin{vmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{vmatrix}. \tag{4.3}$$

The components of the vorticity along the eigenaxis can be found. Thus, the programmer can do far more than the best hot wire or laser anemometer specialist.

Let us examine the three eigenvalues of S. Incompressibility requires that their sum be null. The quantity $\alpha^2+\beta^2+\gamma^2$ is merely a measure of a local time scale. The product is very interesting. If $\alpha\beta\gamma$ is positive, we have one positive and two negative eigenvalues: a fluid element is stretched along one direction and squashed along two others. When $\alpha\beta\gamma$ is negative, the fluid element is deformed as a ball of dough flattened into a pancake. These two cases are illustrated in Fig. 8. We examined the probability dis-

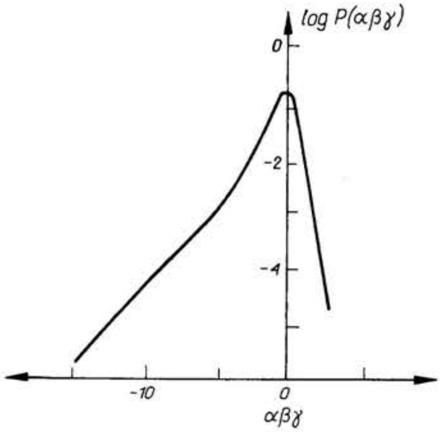

FIG. 7. Probability distribution of the product of the eigen values of the rate of deformation tensor, from a numerical model of turbulence. Deformations which flatten the fluid particles correspond to negatives values of $\alpha\beta\gamma$.

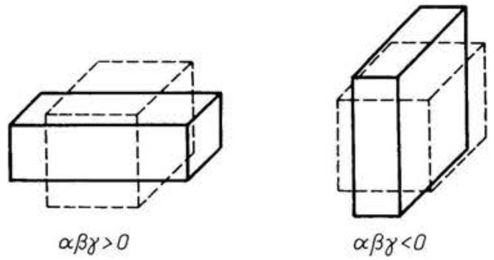

FIG. 8. Deformations for positive and for negative values of $\alpha\beta\gamma$.

On the left: spaghetti type. On the right: pancake type. The initial shape is shown in dotted lines.

tribution of the quantity $\alpha\beta\gamma$. As seen in Fig. 7 it is very assymetric, and practically reduced to a one-sided exponential. Thus, most fluid elements are pancakes, rather than spaghetti. Previous analytic work has anticipated this predominance, but the unbalance of the distribution was unexpected (Ref. 14).

The vorticity can now be examined as it relates to the deformation tensor. The dynamical equations tell us that if the vorticity is perpendicular to the pancake (component $\Omega_\perp$ in Fig. 9), if will be reduced. If it lays in the plane of the pancake ($\Omega_{\parallel}$ in Fig. 9), it will grow. This is the process described first by G. I. Taylor and named "stretching of the vorticity". It corresponds to the conservation of angular momentum in a deformable body. When the momentum of inertia is reduced, the angular velocity increases. We determined, for various time T, the mean square values of $\Omega_\perp$ and of $\Omega_{\parallel}$. The results are shown in Fig. 10. At $T = 0$ and with a factor $1/2$ to account for the fact that $\Omega_{\parallel}$ has twice more components than $\Omega_\perp$, the ratio is unity as it should be. During the early period the vorticity increases moderately, but it turns into the pancakes. Later, the parallel component grows while the perpendicular component decreases. As the energy becomes finally randomized, the situation evolves towards a random vorticity field. This suggests the following model: in a random initial distribution

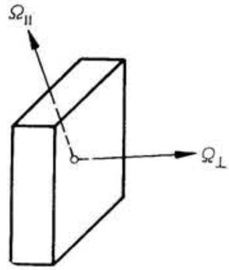

FIG. 9. Vorticity components, in the eigen axis of the rate of deformation tensor, if $\alpha\beta\gamma < 0$.

$\Omega_\perp$ vorticity perpendicular to the pancake, $\Omega_{\parallel}$ vorticity in the pancake.

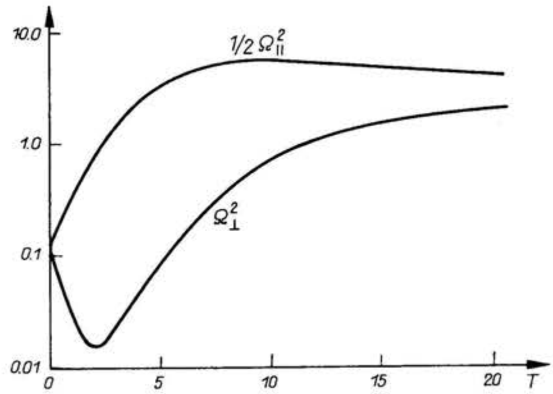

FIG. 10. Evolution of the vorticity, in a numerical model, as it relates to the deformation of fluid elements. Inviscid case.

$\Omega_{\perp}^2$ mean square vorticity perpendicular to pancakes, $\Omega_{\parallel}^2$ mean square vorticity parallel to the pancakes.

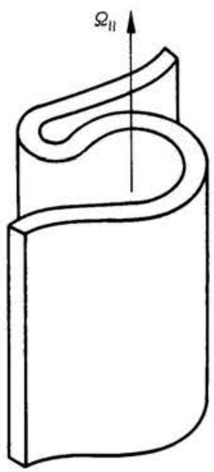

FIG. 11. Effect of concentrated vorticity on a layer of squashed fluid. The pancake is rolled and forms a canelloni.

of velocity, collisions occur between elements moving against each other. The intermediate fluid is squashed, forming a pancake. Its vorticity concentrates in a thin sheet, perpendicular to the direction of collision. The intense vorticity should then roll the pancake and produce something like a canelloni (Fig. 11).

Let us return to the wind tunnel and look at the situation from the point of view of an observer moving with the mean velocity, past the hot-wire probe. If the configuration of Fig. 8 (right side) passes by the anemometer along a line perpendicular to the pancake, the instrument will first sense a velocity below average. As it goes through the pancake, it will rapidly jump to a velocity above average. Thus each pancake produces a positive spike of *du/dt*. Could the clusters of spikes correspond to a canelloni? A firm statement cannot yet be made.

If it is true that physics is a higher form of poetry, our conclusions should be presented as follows:

Big whirls lack smaller whirls,
To feed on their velocity.
They crash and form the finest curls
Permitted by viscosity.

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