66.

A NEW PROOF THAT A GENERAL QUADRIC MAY BE REDUCED TO ITS CANONICAL FORM (THAT IS, A LINEAR FUNCTION OF SQUARES) BY MEANS OF A REAL ORTHOGONAL SUBSTITUTION.

[*Messenger of Mathematics*, XIX. (1890), pp. 1—5.]

ALL the proofs that I am acquainted with (and their name is legion) of the possibility of depriving a quadric, in three or more variables, of its mixed terms by a real orthogonal transformation are made to depend on the theorem that the "*latent roots*" of any symmetrical matrix are all real.

By the latent roots is understood the roots of the determinant expressed by tacking on a variable $-\lambda$ to each term in the diagonal of symmetry to such matrix.

I shall show that the same conclusion may be established *à priori* by purely algebraical ratiocination and without constructing any equation, by the method of cumulative variation. The proof I employ is inductive: that is, if the theorem is true for two or any number of variables I prove that it will be true for one more.

To illustrate the method let us begin with two variables. Consider the form $ax^2 + 2hxy + by^2$.

If in any such form $b = a$, then by an obvious orthogonal transformation, namely, writing $\frac{x+y}{\sqrt{(2)}}$ and $\frac{x-y}{\sqrt{(2)}}$ for x and y, the form becomes

$$a(x^2+y^2) + h(x^2-y^2),$$

or

$$(a+h)x^2 + (a-h)y^2.$$

Now in general on imposing on x, y any orthogonal infinitesimal substitution, so that

$$x \text{ becomes } x + \epsilon y,$$

$$y \quad \text{,,} \quad y - \epsilon x,$$

h in the new form becomes $h + (a - b)\epsilon$, or say $\delta h = (a - b)\epsilon$, and

$$\tfrac{1}{2}\delta(h^2) = (a - b) h\epsilon;$$

the variations of a and b need not be set forth.

Let an infinite succession of such transformations be instituted; then either a and b become equal and the orthogonal substitution above referred to reduces the quadric to its canonical form, in which case this one combined with the preceding infinite series of such substitutions may be compounded into a single substitution, or else by giving ϵ the sign of $(b - a)$ the variation of h^2 may at each step be made negative so that h^2 continually decreases, unless h vanishes. If h does not vanish it must have a minimum value, and this minimum value may be diminished, which involves a contradiction: hence, in the infinite series of substitutions supposed, either a and b become equal or h vanishes, and in either case the quadric is reduced or reducible to its canonical form.

Let us now take the case of three variables x, y, z.

Obviously, by the preceding case, we may make the term involving xy disappear and commence with the initial form

$$ax^2 + by^2 + 2fxz + 2gyz + cz^2.$$

If f or g become zero the quadric may be canonified by virtue of the preceding case.

Again, if $b = a$, by imposing on x, y the orthogonal substitution

$$\frac{g}{\sqrt{(f^2+g^2)}}x + \frac{f}{\sqrt{(f^2+g^2)}}y$$

$$-\frac{f}{\sqrt{(f^2+g^2)}}x + \frac{g}{\sqrt{(f^2+g^2)}}y,$$

the term involving xz will disappear and the final result is the same as if f were zero.

Let us now introduce the infinitesimal orthogonal substitution which changes

$$\begin{array}{lll} x & \text{into} & x + \epsilon y + \eta z, \\ y & ,, & -\epsilon x + y + \theta z, \\ z & ,, & -\eta x - \theta y + z, \end{array}$$

where ϵ, η, θ are supposed to be of the same order of magnitude so that only first powers of them have to be considered.

Then
$$\delta f = (a - c)\eta - g\epsilon,$$
$$\delta g = (b - c)\theta + f\epsilon,$$

also the coefficient of $2xy$ becomes $(a - b)\epsilon - f\theta - g\eta$.

Now whatever η, θ may be, we may determine ϵ in terms of η, θ so that this may be made to vanish, and the initial form of the quadric will be maintained, provided that b is not equal to a.

Hence instituting an infinite series of these infinitesimal substitutions, provided we do not reach a stage where a and b become equal, we may maintain the original form keeping η, θ arbitrary, and shall have

$$\tfrac{1}{2}\delta(f^2+g^2)=(a-c)f\eta+(b-c)g\theta.$$

Suppose a and b to be unequal; therefore $(a-c)$, $(b-c)$ do not vanish simultaneously, and consequently we may make $\delta(f^2+g^2)$ negative unless at least one of the two quantities f, g vanishes.

If neither of them vanishes f^2+g^2 may be made continually to decrease and will have a minimum other than zero, which involves a contradiction.

Hence the infinite series of infinitesimal orthogonal substitutions may be so conducted that either $a-b$ or one at least of the letters f, g shall become zero; and then two additional orthogonal substitutions at most will serve to reduce the Quadric immediately to its canonical form.

I shall go one step further to the case of four variables x, y, z, t and then the course of the induction will become manifest. We may, by virtue of what has been shown, take as our quadric

$$ax^2+by^2+cz^2+2fxt+2gyt+2hzt+dt^2.$$

Here, if any one of the mixed terms disappears, the quadric is immediately reducible by the preceding case, and if any two of the grouped pure coefficients a, b, c become equal (as for instance a, b), then by an orthogonal transformation one of the mixed terms (f or g in the case supposed) may be got rid of; so that this supposition merges in the preceding one.

Impose on x, y, z, t an infinitesimal orthogonal substitution, writing

$$\begin{array}{ll} x+\epsilon y+\theta z+\lambda t & \text{for } x, \\ -\epsilon x+y+\eta z+\mu t & \quad ,, \quad y, \\ -\theta x-\eta y+z+\nu t & \quad ,, \quad z, \\ -\lambda x-\mu y-\nu z+t & \quad ,, \quad t. \end{array}$$

Then

$$\begin{aligned} \delta f &= (a-d)\lambda - g\epsilon - h\theta, \\ \delta g &= (b-d)\mu + f\epsilon - h\eta, \\ \delta h &= (c-d)\nu + f\theta + g\eta. \end{aligned}$$

Also the coefficients of $2xy$, $2xz$, $2yz$ respectively become

$$\begin{aligned} &(a-b)\epsilon - f\mu - g\lambda, \\ &(a-c)\theta - f\nu - h\lambda, \\ &(b-c)\eta - g\nu - h\mu. \end{aligned}$$

Suppose that no two of the grouped pure coefficients a, b, c are equal; then ϵ, θ, η can be, and are to be, expressed in terms of λ, μ, ν so as to make these three expressions vanish; that being done the initial form of the Quadric is maintained throughout the series of substitutions and we may write

$$f\delta f+g\delta g+h\delta h=(a-d)f\lambda+(b-d)g\mu+(c-d)h\nu.$$

Of the three quantities λ, μ, ν it is sufficient for the purpose of the argument to retain any two as λ, μ and to suppose $\nu = 0$.

Then, since we suppose that a and b are not equal,

$$(a-d)f\lambda + (b-d)g\mu$$

(where λ, μ are arbitrary) can always be made negative unless f, g are none of them zero; so that if a and b never become equal *nor* f or g vanish $f^2 + g^2 + h^2$ cannot have any minimum value other than zero, which involves a contradiction; hence in the course of the series of infinitesimal transformations either a and b must become equal, or f or g or both of them vanish. If f and g vanish simultaneously or even if one only of them vanish, then *one* succeeding substitution, and if a and b become equal *two* succeeding substitutions, will effect the reduction to the canonical form. This proves the theorem for four variables.

The method is obviously extendible to any number of variables; in the case just considered it is seen that in the infinitesimal orthogonal matrix of substitution for the *exceptional* line or column (that which relates to the *excepted* variable the t) it is not necessary to employ more than two arbitrary infinitesimals and a like remark applies to the general case, so that if there are n variables, whilst $\frac{1}{2}(n^2 - n)$ is the number of infinitesimals that would appear in the complete matrix, $\frac{1}{2}(n^2 - 3n + 6)$, that is $\frac{1}{2}\{(n-1)(n-2)\} + 2$, are sufficient for the purpose of the demonstration.

Thus then without recourse to any theorem of Equations it is proved that any Quadric may be reduced by a real orthogonal substitution to its canonical form*.

* I have applied the same method to prove that by two real independent orthogonal substitutions operated on

$$x_1, x_2, \dots x_n; \; y_1, y_2, \dots y_n$$

the general lineo-linear Quantic in the x's and y's (with real coefficients) may be reduced to the canonical form $\Sigma x_i y_i$, and have sent for insertion in the *Comptes Rendus* of the Institute a Note in which I give the rule for effecting this reduction [above, p. 638].

It may be sufficient here to mention that if U is the given lineo-linear Quantic, its n canonical multipliers are the square roots of the n canonical multipliers of the Quadric $\Sigma\left(\frac{dU}{dy}\right)^2$, or if we please of $\Sigma\left(\frac{dU}{dx}\right)^2$, which it may easily be shown *à posteriori* are necessarily omni-positive; and I need hardly add that although these two Quadrics are different, their canonical multipliers are the same.