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ON THE THEORY OF GROUPS, AS DEPENDING ON THE SYMBOLIC EQUATION $\theta^n = 1$.—SECOND PART.

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IMAGINE the symbols

$$L, M, N, \ldots$$

such that (L being any symbol of the system),

$$L^{-1}L, L^{-1}M, L^{-1}N, \ldots$$

is the *group*

$$1, \alpha, \beta, \ldots;$$

then, in the first place, M being any other symbol of the system, $M^{-1}L$, $M^{-1}M$, $M^{-1}N, \ldots$ will be the same group $1, \alpha, \beta, \ldots$. In fact, the system $L, M, N, \ldots$ may be written $L, L\alpha, L\beta \ldots$; and if e.g. $M = L\alpha$, $N = L\beta$ then

$$M^{-1}N = (L\alpha)^{-1} L\beta = \alpha^{-1} L^{-1} L\beta = \alpha^{-1}\beta,$$

which belongs to the group $1, \alpha, \beta, \ldots$.

Next it may be shown that

$$LL^{-1}, ML^{-1}, NL^{-1}, \ldots$$

is a group, although not in general the same group as $1, \alpha, \beta, \ldots$. In fact, writing $M = L\alpha$, $N = L\beta$, &c., the symbols just written down are

$$LL^{-1}, L\alpha L^{-1}, L\beta L^{-1}, \ldots$$

and we have e.g. $L\alpha L^{-1} . L\beta L^{-1} = L\alpha\beta L^{-1} = L\gamma L^{-1}$, where γ belongs to the group $1, \alpha, \beta$.

The system $L, M, N, \ldots$ may be termed a group-holding system, or simply a holder; and with reference to the two groups to which it gives rise, may be said to hold on the nearer side the group $L^{-1}L, L^{-1}M, L^{-1}N, \ldots$, and to hold on the further side the group $LL^{-1}, LM^{-1}, LN^{-1}, \ldots$ Suppose that these groups are one and the same group $1, \alpha, \beta \ldots$, the system $L, M, N, \ldots$ is in this case termed a symmetrical holder, and in reference to the last-mentioned group is said to hold such group symmetrically. It is evident that the symmetrical holder $L, M, N, \ldots$ may be expressed indifferently and at pleasure in either of the two forms $L, L\alpha, L\beta, \ldots$ and $L, \alpha L, \beta L$; i.e. we may say that the group is convertible with any symbol L of the holder, and that the group operating upon, or operated upon by, a symbol L of the holder, produces the holder. We may also say that the holder operated upon by, or operating upon, a symbol α of the group reproduces the holder.

Suppose now that the group

$$1, \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \ldots$$

can be divided into a series of symmetrical holders of the smaller group

$$1, \alpha, \beta, \ldots;$$

the former group is said to be a multiple of the latter group, and the latter group to be a submultiple of the former group. Thus considering the two different forms of a group of six, and first the form

$$1, \alpha, \alpha^2, \gamma, \gamma\alpha, \gamma\alpha^2, (\alpha^3 = 1, \gamma^2 = 1, \alpha\gamma = \gamma\alpha),$$

the group of six is a multiple of the group of three, $1, \alpha, \alpha^2$ (in fact, $1, \alpha, \alpha^2$ and $\gamma, \gamma\alpha, \gamma\alpha^2$ are each of them a symmetrical holder of the group $1, \alpha, \alpha^2$); and so in like manner the group of six is a multiple of the group of two, $1, \gamma$ (in fact, $1, \gamma$ and $\alpha, \alpha\gamma$, and $\alpha, \alpha^2\gamma$ are each a symmetrical holder of the group $1, \gamma$). There would not, in a case such as the one in question, be any harm in speaking of the group of six as the product of the two groups $1, \alpha, \alpha^2$ and $1, \gamma$, but upon the whole it is, I think, better to dispense with the expression.

Considering, secondly, the other form of a group of six, viz.

$$1, \alpha, \alpha^2, \gamma, \gamma\alpha, \gamma\alpha^2 (\alpha^3 = 1, \gamma^2 = 1, \alpha\gamma = \gamma\alpha^2);$$

here the group of six is a multiple of the group of three, $1, \alpha, \alpha^2$ (in fact, as before, $1, \alpha, \alpha^2$ and $\gamma, \gamma\alpha, \gamma\alpha^2$, are each a symmetrical holder of the group $1, \alpha, \alpha^2$, since, as regards $\gamma, \gamma\alpha, \gamma\alpha^2$, we have $(\gamma, \gamma\alpha, \gamma\alpha^2) = \gamma(1, \alpha, \alpha^2) = (1, \alpha^2, \alpha)\gamma$). But the group of six is not a multiple of any group of two whatever; in fact, besides the group $1, \gamma$ itself, there is not any symmetrical holder of this group $1, \gamma$; and so, in like manner, with respect to the other groups of two, $1, \gamma\alpha$, and $1, \gamma\alpha^2$. The group of three, $1, \alpha, \alpha^2$, is therefore, in the present case, the only submultiple of the group of six.

It may be remarked, that if there be any number of symmetrical holders of the same group, $1, \alpha, \beta, \ldots$ then any one of these holders bears to the aggregate of the holders a relation such as the submultiple of a group bears to such group; it is proper to notice that the aggregate of the holders is not of necessity itself a holder.