110.

NOTE ON THE TRANSFORMATION OF A TRIGONOMETRICAL EXPRESSION.

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THE differential equation

$$\frac{dx}{(a+x)\sqrt{(c+x)}}+\frac{dy}{(a+y)\sqrt{(c+y)}}+\frac{dz}{(a+z)\sqrt{(c+z)}}=0,$$

integrated so as to be satisfied when the variables are simultaneously infinite, gives by direct integration

$$\tan^{-1}\sqrt{\left(\frac{a-c}{c+x}\right)}+\tan^{-1}\sqrt{\left(\frac{a-c}{c+y}\right)}+\tan^{-1}\sqrt{\left(\frac{a-c}{c+z}\right)}=0\,;$$

and, by Abel's theorem,

$$\left|\begin{matrix} 1, & x, & (a+x)\sqrt{(c+x)} \\ 1, & y, & (a+y)\sqrt{(c+y)} \\ 1, & z, & (a+z)\sqrt{(c+z)} \end{matrix}\right|=0.$$

To show *à posteriori* the equivalence of these two equations, I represent the determinant by the symbol $\square$, and expressing it in the form

$$\square=\left|\begin{matrix} 1, & a+x, & (a+x)\sqrt{(c+x)} \\ \vdots & & \end{matrix}\right|,$$

I write for the moment $\xi=\sqrt{\left(\frac{a-c}{c+x}\right)}$ &c.; this gives

$$\square = \left| \begin{array}{lll} 1, & (a-c)\left(1+\frac{1}{\xi^2}\right), & (a-c)^{\frac{3}{2}}\left(\frac{1}{\xi}+\frac{1}{\xi^3}\right) \\ \vdots & & \end{array} \right|$$

$$= \frac{(a-c)^{\frac{5}{2}}}{\xi^3\eta^3\zeta^3} \left| \begin{array}{lll} \xi^3, & \xi^3+\xi, & \xi^2+1 \\ \vdots & & \end{array} \right|$$

$$= \frac{(a-c)^{\frac{5}{2}}}{\xi^3\eta^3\zeta^3} \left| \begin{array}{lll} \xi^3, & \xi, & \xi^2+1 \\ \vdots & & \end{array} \right|$$

$$= -\frac{(a-c)^{\frac{5}{2}}}{\xi^3\eta^3\zeta^3} \left\{ \left| \begin{array}{lll} 1, & \xi, & \xi^3 \\ \vdots & & \end{array} \right| - \xi\eta\zeta \left| \begin{array}{lll} 1, & \xi, & \xi^2 \\ \vdots & & \end{array} \right| \right\}$$

$$= -\frac{(a-c)^{\frac{5}{2}}}{\xi^3\eta^3\zeta^3} (\xi+\eta+\zeta-\xi\eta\zeta) \left| \begin{array}{lll} 1, & \xi, & \xi^2 \\ \vdots & & \end{array} \right|;$$

or, replacing ξ, η, ζ by their values, we have identically

$$\left| \begin{array}{lll} 1, & x, & (a+x)\sqrt{(c+x)} \\ 1, & y, & (a+y)\sqrt{(c+y)} \\ 1, & z, & (a+z)\sqrt{(c+z)} \end{array} \right| =$$

$$-\frac{(c+x)^{\frac{3}{2}}(c+y)^{\frac{3}{2}}(c+z)^{\frac{3}{2}}}{(a-c)^2} \left\{ \sqrt{\frac{a-c}{c+x}} + \sqrt{\frac{a-c}{c+y}} + \sqrt{\frac{a-c}{c+z}} - \sqrt{\frac{a-c}{c+x}}\sqrt{\frac{a-c}{c+y}}\sqrt{\frac{a-c}{c+z}} \right\} \left| \begin{array}{lll} 1, & \sqrt{\frac{a-c}{c+x}}, & \frac{a-c}{c+x} \\ 1, & \sqrt{\frac{a-c}{c+y}}, & \frac{a-c}{c+y} \\ 1, & \sqrt{\frac{a-c}{c+z}}, & \frac{a-c}{c+z} \end{array} \right|,$$

and the equation

$$\sqrt{\frac{a-c}{c+x}} + \sqrt{\frac{a-c}{c+y}} + \sqrt{\frac{a-c}{c+z}} - \sqrt{\frac{a-c}{c+x}}\sqrt{\frac{a-c}{c+y}}\sqrt{\frac{a-c}{c+z}} = 0$$

is of course equivalent to the trigonometrical equation

$$\tan^{-1}\sqrt{\frac{a-c}{c+x}} + \tan^{-1}\sqrt{\frac{a-c}{c+y}} + \tan^{-1}\sqrt{\frac{a-c}{c+z}} = 0,$$

which shows the equivalence of the two equations in question.