32.

ON THE SOLUTION OF A CLASS OF EQUATIONS IN QUATERNIONS.

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THE general equation of the degree ω in Quaternions or Binary Matrices is obviously ω^4, but in certain cases some of these roots evaporate and go off to infinity. The only equation considered by Sir William Hamilton in his Lectures is the Quadratic Equation of a form which I call unilateral, because the quaternion coefficients in it are supposed all to lie on the same side of the unknown quantity. I propose here to show how Hamilton's equation, and indeed a unilateral one of any order, may be solved by a general algebraical method and the number of its roots determined.

It will be convenient to begin by setting out certain general equations relating to any two binary matrices m, n.

Writing the determinant of $x+ym+zn$ under the form

$$x^2+2bxy+2cxz+dy^2+2eyz+fz^2$$

(b, c, d, e, f, thus constituting what I call the parameters of the *corpus* m, n), we have universally

$$m^2-2bm+d=0, \quad n^2-2cn+f=0, \quad d(m^{-1}n)^2-2e(m^{-1}n)+f=0.$$

Moreover if m, n receive the scalar increments μ, ν; d, e, f become respectively

$$d-2\mu b+\mu^2, \quad e-\mu c-\nu b+\mu\nu, \quad f-2\nu c+\nu^2.$$

Let us begin with Hamilton's form, say

$$x^2-2px+q=0,$$

and suppose

$$x^2-2Bx+D=0,$$

where B, D are scalars to be determined.

Let b, c, d, e, f be the five known parameters of the *corpus* p, q. Then, since

$$(p-B)^{-1}(q-D)=2x,$$

we shall have [cf. p. 188 above]

$$4(d-2bB+B^2)x^2-4(e-bD-cB+BD)x+f-2cD+D^2=0.$$

Hence, writing $B-b=u,\quad D-c=v,$

$$d-b^2=\alpha,\quad e-bc=\beta,\quad f-c^2=\gamma,$$

we have $u^2+\alpha=\lambda,\quad uv+\beta=2\lambda(u+b),\quad v^2+\gamma=4\lambda(v+c).$

From the last two equations, eliminating v, there results

$$(2\lambda u-2b\lambda-\beta)^2-4\lambda(2\lambda u-2b\lambda-\beta)u+(\gamma-4c\lambda)u^2=0.$$

Hence substituting $\lambda-\alpha$ for u^2,

$$(4\lambda^2+4c\lambda-\gamma)(\lambda-\alpha)-(2b\lambda-\beta)^2=0.$$

We have thus six values of u, namely

$$\pm\sqrt{(\lambda-\alpha)}$$

(where λ has three values), to which correspond six values of v, namely

$$2\lambda\pm\frac{2\lambda b-\beta}{\sqrt{(\lambda-\alpha)}};$$

and, finally, $2x=(p-u-b)^{-1}(q-v+c)$

$$=\{(p-b)^2-u^2\}^{-1}(p-b+u)(q-c-v),$$

or

$$x=\frac{pq-(c+v)p-(b-u)q+(b-u)(c+v)}{2(b^2-d-u^2)};$$

which equation gives six values for x, and shows that ten have evaporated.

It is easy to account *à priori* for the solution depending only upon a cubic in u^2.

For $x^2-2px+q=0$ is the same as $y^2-2yp+q=0$, where $y=-x+2p$. But obviously, from the nature of the process for determining them, B and C are independent of the *side* of the unknown on which the first coefficient lies. Hence the actual B will be associated with B', B' being what B becomes when x becomes $-x+2p$, which is obviously $-B+2b$.

Hence with any value of $B-b$, which is u, is associated a corresponding $B-b$, which is $-u$.

I will now proceed to apply a similar or the same method to the trinomial cubic equation in quaternions (or binary quantity) $x^3+px-q=0$, with a view to ascertain the number of its roots.

Retaining the same notation as before, and still supposing

$$x^2-2Bx+D=0,$$

we obtain $$x^3 + (D - 4B^2)x + 2BD = 0,$$

and $$x = \frac{:q + 2BD}{p + 4B^2 - D}.^{*}$$

Hence $$\{(4B^2 - D)^2 - 2b(4B^2 - D) + d\}x^2$$
$$-2\{2(4B^2 - D)BD - c(4B^2 - D) - 2bBD + e\}x$$
$$+ 4B^2D^2 - 2cBD + f = 0.$$

Hence we may write

$$(4B^2 - D)^2 - 2b(4B^2 - D) + d = \lambda,$$
$$2(4B^2 - D)BD - c(4B^2 - D) - 2bBD + e = \lambda B,$$
$$4B^2D^2 - 2cBD + f = \lambda D;$$

from which equations B and D are to be determined. Eliminating λ between the first and second and between the first and third of these equations, we obtain two equations, of which the arguments are

$$D^3;\quad B^2D^2, D^2;\quad B^4D, B^2D, BD, D;\quad 1$$

for the one,

$$BD^2;\quad B^3D, BD, D;\quad B^5, B^3, B^2, B;\quad 1$$

for the other.

Eliminating D by the Dialytic method between these two equations, we shall have (using points to signify unexpressed coefficients) the following three linear equations in D^2, D, 1, namely:

$$\cdot BD^2 + (\cdot B^3 + \&c.)D + (\cdot B^5 + \&c.) = 0,$$
$$\cdot B^3D^2 + (\cdot B^5 + \&c.)D + (\cdot B^7 + \&c.) = 0,$$
$$\cdot B^5D^2 + (\cdot B^7 + \&c.)D + (\cdot B^9 + \&c.) = 0.$$

Hence in the final equation B rises to the 15th power; and by combining any two of the above equations, D is given linearly in terms of B; and, finally, x is known from the equation

$$x = \frac{:(p + D - 4B^2 - 2b)(q + 2BD)}{-(4B^2 - D)^2 - 2(4B^2 - D) + d},$$

and has 15 values.

A like process may be extended to a unilateral equation (of the Jerrardian form) of any degree, say $x^\omega + qx + r = 0$.

Introducing the auxiliary equation with scalar coefficients as before, namely

$$x^2 - 2Bx + D = 0,$$

x may be expressed as a function of q, r, B, D; and the term containing the

* I use $\frac{:L}{M}$ and $\frac{L:}{M}$ to signify $M^{-1}L$ and LM^{-1} respectively.

highest power of B in the equation for determining B (of which D is a one-valued function), when $\omega = 4$, will be found to be the determinant

$$\begin{array}{cccc} \cdot B & \cdot B^3 & \cdot B^5 & \cdot B^7 \\ \cdot B^3 & \cdot B^5 & \cdot B^7 & \cdot B^9 \\ \cdot B^5 & \cdot B^7 & \cdot B^9 & \cdot B^{11} \\ \cdot B^7 & \cdot B^9 & \cdot B^{11} & \cdot B^{13} \end{array} *$$

and a similar determinant will fix the degree of B in the resolving equation for any value of ω. Hence the number of solutions of the unilateral equation in quaternions of the Jerrardian form of the degree ω is $\omega(2\omega - 1)$ or $2\omega^2 - \omega$, and the evaporation will accordingly be $\omega^4 - 2\omega^2 + \omega$, or

$$(\omega^2 - \omega)(\omega^2 + \omega - 1).$$

Moreover the same method with a slight addition will serve to determine the roots of the general unilateral equation in quaternions, the number of which will be a cubic function of ω, as I propose to show and to give its precise value in some future communication, either in this Journal, or at all events in the memoir on Universal Algebra now in the course of publication, under the form of lectures, in the *American Journal of Mathematics*†.

I very much question whether the old method of Hamilton, as taught by its most consummate masters, Tait in this country, or the late Prof. Benjamin Peirce in America, would be found sufficiently plastic to deal effectually with an analytical investigation in quaternions of this degree of complexity, so as to lead to the formula for the number of solutions of the unilateral equation of the Jerrardian form above given.

I invite my much esteemed and most capable former colleague and former pupil, Dr Story, of the Johns Hopkins, and Prof. Stringham, of the University of California, who carry on the traditions of the Harvard School, to put the power of the old method as compared with the new to this practical test.

Postscript.—If $x^3 - 3px^2 + 3qx - r = 0$,

(where p, q, r are perfectly general matrices of the second order which satisfy the general equations

$$q^2 - 2bq + d = 0, \quad qr + rq - 2bq - 2b_1q + 2\epsilon = 0, \quad r^2 - 2b_1q + d_1 = 0,$$
$$pq + qp - 2bp - 2\beta q + 2e = 0, \quad p^2 - 2\beta p + \delta = 0,$$
$$pr + rp - 2b_1p - 2\beta r + 2e_1 = 0),$$

* It may readily be seen that the highest term in the equation for finding B is identical with the resultant of

$$D^4 - 24B^2D^3 + 80B^4D^2 \text{ and } 4BD^3 - 40B^3D^2 + 64B^5D - 64B^7,$$

that is, will be $2^{18}.3.7.19B^{28}$; and that the last term (at all events to the sign *près*) will be $b^4\delta^2$, which is of $4.3 + 2.2.4$ (that is of 28) dimensions in x, and is therefore codimensional (as it ought to be) with B^{28}.

† It is given in the Postscript below.

and if we write $$x^2 - 2Bx + D = 0,$$

then $$px = \frac{:r + 3Dp - BD}{3q - 3Bp + B^2 - D};$$

and I find by perfectly easy and straightforward work that B, D may be determined by means of the following equations:

$$\frac{(B^2-D)^2}{9} + 2(b - \beta B)\frac{B^2 - D}{3} + (d - 2eB + 4\delta B^2) = 9\lambda,$$

$$\frac{B^3 D - BD^2}{3} + (b_1 + 3\beta D)\frac{B^2 - D}{3} + (\epsilon - e_1 B + 3eD - 6\delta BD) = 3B\lambda,$$

$$B^2 D^2 - 2(b_1 + 3\beta D) BD + d_1 + 6De_1 + 9\delta D^2 = D\lambda.$$

The order (by which I mean the number of solutions of this system of equations) is readily seen to be the same as that of

$$\cdot D^3 \;+ \cdot B^3 D + \cdot B^4 D = 0$$
$$\cdot BD^2 + \cdot B^3 D + \cdot B^5 \;= 0;$$

that is, is the same as the degree in B of $B^3(B^5)^2 . R$, where R is the resultant of

$$\cdot D^2 + \cdot B^3 + \cdot B^4 \text{ and } \cdot D^2 + \cdot B^2 D + \cdot B^4.$$

Hence* the number of solutions is $3 + 10 + 8$, that is, is 21.

Practically, therefore, we have now sufficient data to determine the number of solutions of a unilateral equation in quaternions of any order ω; for it is morally certain that such number is a rational function of ω; and as it cannot but be of a lower order than ω^4, we have only to determine a cubic function of ω whose values for $\omega = 0, 1, 2, 3$ are 0, 1, 6, 21, which is easily found to be $\omega^3 - \omega^2 + \omega$; so that the evaporation is $\omega^4 - \omega^3 + \omega^2 - \omega$, that is

$$(\omega^2 + 1)(\omega^2 - \omega).$$

Practically also we can solve (subject to hardly needful verification) the number of roots of a unilateral equation of the special form

$$x^\omega + q_\theta x^\theta + q_{\theta-1} x^{\theta-1} + \ldots + q_0 = 0.$$

For when $\theta = \omega$, we know the number is ω^2; and when $\theta = 1$, the number is $\omega^3 + \omega^2 - \omega$; consequently if the second differences of the function of (ω, θ) which expresses the number of roots are constant, the value of this function when $\theta = \omega - 1$ is $\omega^3 - \omega^2 + \omega$, which we have found to be the actual number; and consequently, if the second differences are not constant, they must be sometimes positive and sometimes negative, which is in the highest degree improbable. Hence in all probability it will be found that the required number of solutions in the form supposed is $(1 + \theta)\omega^2 - \theta\omega$.

I need hardly add that the nine quantities $2b$, $2b_1$, 2β; $2e$, $2e_1$, 2ϵ; d, δ, d_1, which occur in the discussion above given of the general unilateral cubic, or, say, rather the ten quantities obtained by adding on to these *unity*, are the

[* See footnote † p. 197 above.]

ten coefficients of the determinant to the binary matrix $(x+py+qz+rt)$, which of course there is not the slightest difficulty in expressing in terms of scalar and vector affections of p, q, r and their combinations, if any one chooses to regard them as given in quaternion form.

Scholium. In what precedes it is very requisite to notice that only *general* cases are considered; and that there are multitudinous others which escape the direct application of this method, and do not conform to the rule which assigns the number of solutions. Thus, for example, the equation $x^2+px=0$, besides the solutions $x=0$, $x=-p$, will have two others which will require the method of the text to be modified in order to determine. Or take the most elementary case of all, the simple equation $px=q$. If p is not vacuous (that is, if its determinant when regarded as a matrix, or its modulus when regarded as a quaternion, is finite), there is the one solution $x=p^{-1}q$. But if p is vacuous, then, unless q is also vacuous, the equation is insoluble. If $q=0$, there will be two solutions; one of them $x=0$, the other $x=$ conjugate of p in quaternion terminology; or

$$x=\begin{matrix}-d; & b\\ c; & -a\end{matrix}, \text{ when } p=\begin{matrix}a; & b\\ c; & d\end{matrix}$$

in the language of matrices. If, p still remaining vacuous, q is vacuous but not zero, a further condition must be satisfied, namely, if

$$p=\begin{matrix}a; & b\\ c; & d\end{matrix} \text{ and } q=\begin{matrix}\alpha; & \beta\\ \gamma; & \delta\end{matrix},$$

the condition is $$a\delta+\alpha d-b\gamma-c\beta=0;$$

or if $$p=a+bi+cj+dk \text{ and } q=\alpha+\beta i+\gamma j+\delta k,$$

the condition is $$a\alpha+b\beta+c\gamma+d\delta=0.$$

When this condition (besides that of q being vacuous) is satisfied, the equation $px=q$ is soluble, and $p^{-1}q$ becomes finite but indeterminate, containing two arbitrary constants*.

* So in general if p, q be two simply vacuous matrices of any order, the condition that the equation $px=q$ may be soluble, or, in other words, that $p^{-1}q$ (a combination of an ideal with a vacuous matrix) may be non-ideal, may be shown to be that the determinant to the matrix $\lambda p+\mu q$ (where λ, μ are scalar quantities) shall vanish identically—which (p being supposed already to be vacuous) involves just as many additional conditions as there are units in the order of the matrix.