15.

ON INVOLUTANTS AND OTHER ALLIED SPECIES OF INVARIANTS TO MATRIX SYSTEMS.

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To make what follows intelligible I must premise the meaning and laws of vacuity and nullity.

A matrix is said to be vacuous when its content (the determinant of the matrix) is zero, but it may have various degrees of vacuity from 0 up to ω the order of the matrix.

If from each term in the principal diagonal of a matrix λ be subtracted, the content of the resulting matrix is a function of degree ω in λ; the ω values of λ which make this content vanish are called its latent roots, and if i of these roots are zero, the vacuity (treated as a number) is said to be i. This comes to the same thing as saying that the vacuity is i when the determinant, and the sums of the determinants of the principal minors of the orders $\omega-1$, $\omega-2$, ... $(\omega-i+1)$ are each zero. A principal minor of course means one which is divided into 2 [equal] triangles by the principal diagonal of the parent matrix.

Again the nullity is said to be i when *every* minor of the order $(\omega-i+1)$, and consequently of each superior order, is zero. It follows therefore that it means the same thing to predicate a vacuity 1 and a nullity 1 of any matrix, but for any value of i greater than 1, a nullity i implies a vacuity i but not *vice versâ*; the vacuity may be i, whilst the nullity may have any value from 1 up to i inclusive.

The law of nullity which I am about to enunciate is one of paramount importance in the theory of matrices*.

* The three cardinal laws or landmarks in the science of multiple quantity are (1) the law of *nullity*, (2) the law of *latency*, namely, that if $\lambda_1, \lambda_2, \ldots \lambda_\omega$ are the latent roots of m, then $f\lambda_1, f\lambda_2, \ldots f\lambda_\omega$ are those of fm, including as a consequence that

$$fm = \Sigma f\lambda_1 \frac{(m-\lambda_2)(m-\lambda_3)\ldots(m-\lambda_\omega)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_\omega)},$$

and (3) the law of *identity*, namely, that the powers and combinations of powers of two matrices m, n of the order ω are connected together by $(\omega+1)$ equations whose coefficients are all included among the coefficients of the determinant to the Matrix

$$x+ym+zn.$$

The law is that the nullity of the product of *two* (and therefore of any number of) matrices cannot be less than the nullity of any factor nor greater than the sum of the nullities of the several factors which make up the product.

Suppose now that $\lambda_1, \lambda_2, \ldots \lambda_\omega$ are the latent roots of any matrix with unequal latent roots of the order ω. It is obvious that any such term as $m - \lambda_1$ will have the nullity 1, for its latent roots will be 0, $\lambda_2 - \lambda_1$, $\lambda_3 - \lambda_1, \ldots \lambda_\omega - \lambda_1$, and consequently its *vacuity* is 1.

Moreover we know from Cayley's famous identical equation that the nullity of the product of all the ω factors is ω.

Hence it follows that if M_i contains i, and M_j the remaining $\omega - i$ of these factors (so that $i + j = \omega$), the nullity of M_i must be exactly i and of M_j exactly j.

For the theorem above stated shows that M_i cannot have a nullity greater than i, nor M_j a nullity greater than j.

Hence if the nullity of the one were less than i or of the other less than j, the nullity of $M_i M_j$ would be less than $i + j$, that is, less than ω, whereas its nullity is ω; hence the two nullities are respectively i and j as was to be shown.

Furthermore we know that the latent roots of $(m - \lambda_1)^\alpha$ are $(\lambda_1 - \lambda_1)^\alpha$; $(\lambda_2 - \lambda_1)^\alpha$; ... $(\lambda_\omega - \lambda_1)^\alpha$.

Hence if the latent roots of m are all distinct, the nullity of $(m - \lambda_1)^\alpha$ is unity and consequently by the same reasoning as that above employed it follows that the nullity of

$$(m - \lambda_1)^{\alpha_1} . (m - \lambda_2)^{\alpha_2} \ldots (m - \lambda_i)^{\alpha_i}$$

is exactly i.

I will now explain what is meant by the Involutant or Involutants of a system of two matrices of like order.

It will be convenient here to introduce the term "topical resultant" of a system of ω^2 matrices each of order ω.

We may denote any matrix say

$$\begin{matrix} a_{1,1} & a_{1,2} & \cdots & a_{1,\omega} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,\omega} \\ \cdots & \cdots & \cdots & \cdots \\ a_{\omega,1} & a_{\omega,2} & \cdots & a_{\omega,\omega} \end{matrix}$$

by the linear form

$$\begin{aligned} & a_{1,1} t_{1,1} + a_{1,2} t_{1,2} + \ldots + a_{1,\omega} t_{1,\omega} \\ + & a_{2,1} t_{2,1} + a_{2,2} t_{2,2} + \ldots + a_{2,\omega} t_{2,\omega} \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + & a_{\omega,1} t_{\omega,1} + a_{\omega,2} t_{\omega,2} + \ldots + a_{\omega,\omega} t_{\omega,\omega} \end{aligned}$$

where the t system is the same for all matrices of the order ω. If, then, we have ω^2 such matrices, their topical resultant is the Resultant in the ordinary sense of the ω^2 linear forms above written, proper to each of them respectively.

Suppose now that m, n are two independent matrices of the order ω, we may form ω^2 matrices by taking each power of m from 0 to $\omega - 1$ as an antecedent factor, and can combine it with similar powers of n as a consequent factor, and in this way obtain ω^2 matrices, of which the first will be the ω-ary unity, that is, a matrix of the order ω in which the principal diagonal terms are all units and the other terms all zero. The topical resultant of these ω^2 matrices I shall for brevity denote as the Involutant to m, n.

In like manner, inverting the position of the powers of m and of n so as to make the latter precede instead of following the former in the ω^2 products above referred to, we shall obtain another topical resultant which may be termed the Involutant to n, m.

The reason why I speak of these topical resultants as involutants to m, n or n, m is the following:

In general if m, n are two independent matrices, any other matrix p, by means of solving ω^2 linear equations, may obviously be expressed as a linear function of the ω^2 products

$$(1, m, m^2, \ldots, m^{\omega-1})(1, n, n^2, \ldots, n^{\omega-1}).$$

There are, however, exceptions to this fact.

The most obvious exception is that which takes place when n is a function of m; for then any ω of the ω^2 products will be linearly related, and there will be substantially only ω disposable quantities to solve ω^2 equations.

Another exception is when the m, n Involutant, that is, the topical resultant of the ω^2 matrices, is zero; in which case the general values of the ω^2 disposable quantities each becomes infinite. So that m, n may be said to be in a kind of mutual involution with one another. So, again, p may in general be expressed as a linear function of the ω^2 matrices

$$(1, n, n^2, \ldots, n^{\omega-1})(1, m, m^2, \ldots, m^{\omega-1}),$$

but when the n, m Involutant vanishes this is no longer possible.

When $\omega = 2$ the two involutants, considered as definite determinants, are absolutely equal in magnitude and in Algebraical sign, but when ω exceeds 2 this is no longer the case; the two Involutants are then entirely distinct functions of the elements of m and n.

Thus to take a simple example: if $m = \begin{matrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{matrix}$ and $n = \begin{matrix} 0 & \rho & k \\ k & 0 & \rho^2 \\ 1 & k & 0 \end{matrix}$ it will be found by direct calculation of two topical resultants of the 9th order, that the two involutants will be

$$81(\rho - \rho^2)(k^3 - \rho)^3 \text{ and } 81(\rho^2 - \rho)(k^3 - \rho^2)^3$$

respectively. The reason why the two involutants coincide in the case of $\omega = 2$ is not far to seek. It depends upon the fact of the existence of the mixed identical equation

$$mn + nm - 2bn - 2cm + 2e = 0;$$

from which it is obvious that the topical resultant of 1, m, n, mn is the negative of that of 1, m, n, nm or identical with that of 1, n, m, nm.

By direct calculation it will be found that the Involutant m, n, or n, m, where $m = \begin{matrix} f & g \\ h & k \end{matrix}$ $n = \begin{matrix} f' & g' \\ h' & k' \end{matrix}$ is

$$-(gh' - g'k)^2 + \{(f-k)g' - (f'-k')g\}\{(f-k)h' - (f'-k')h\},$$

which is the same thing as the content of the matrix $(mn - nm)$. It may also be shown *à priori* or by direct comparison to be identical (to a numerical factor *près*) with the Discriminant of the Determinant to the matrix $(x + ym + zn)$ which is a ternary quantic of the second order. Its actual value is 4 times that discriminant.

Let us consider the analogous case of Mechanical Involution of lines in a plane or in space. There are two questions to be solved. The one is to find the condition that the Involution may exist, that is, that a set of equilibrating forces admit of being found to act along the lines; the second, to determine the relative magnitudes of the forces when the involution exists, and this is the simpler question of the two.

In like manner we may consider two questions in the case of m, n being in either of the two kinds of involution; the one being to find what the condition is of such involution existing, the other what are the coefficients of the ω^2 coefficients in the equation which connects the ω^2 products, when the involution exists.

This latter part of the question (surprising as the assertion may appear and is) admits of a very simple and absolutely general direct and almost instantaneous solution by means of the Law of Nullity, above referred to, as I will proceed to show.

The determination of the Involutants, or at all events of their product, will then be seen to follow as an immediate consequence from this prior determination of the form of the equations which express the involutions of the two kinds respectively.

But first it may be well to explain why and in what sense I refer in the title to Involutants as belonging to a class of invariants. I say, then, that universally involutants are invariants in this sense, that if for m and for n, any function of m, or any function of n be substituted, the ratio of the two Involutants, say I and J, remains unaltered. By virtue of the Identical Equation $(m)^i$ will be of the form of

$$A_i + B_i + C_i m^2 + \ldots + L_i m^{\omega-1}$$

and as a consequence it is easy to see that when m^i is substituted for m, I and J will become respectively PI, PJ where P is the ωth power of the determinant to the matrix formed by writing under one another the $(\omega - 1)$ lines of terms, of which the line $B_i, C_i, \ldots; L_i$ is the general expression.

Moreover, in the particular case where $\omega = 2$ and $I = J$*, besides being an Invariant in this modified sense, I will be an invariant in a sense including but transcending the more ordinary conception of an Invariant; for if when, for m and n, $f(m, n)$ and $\phi(m, n)$ are substituted, I becomes I', then I' will contain I as a factor; this is a consequence of the fact that when m and n are in involution $f(m, n)$ and $\phi(m, n)$ will also be in involution, for in consequence of the identical equation

$$mn + nm - 2bn - 2cm + 2e = 0$$

f and ϕ and $f\phi$ will each be reducible to the form

$$A + Bm + Cn + Dmn$$

and it is obvious from the ordinary theory of the determinants that the topical resultant of 1, $\left(\text{meaning } \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right)$, and three linear functions of 1, m, n, nm, will contain as a factor the topical resultant of 1, m, n, mn.

Nor must it be supposed that Involutants are the only species of invariants in the modified sense first described which appertain to the

* I for some time had imagined, and indeed thought I had proved, that the two involutants were always identical. When crossing the Atlantic last month on board the "Arizona," having hit upon a pair of matrices of the third order, for which the two topical resultants admitted of easy calculation, I found, to my surprise, that they were perfectly distinct. The cause of the failure of the supposed proof constitutes a paradox which will form the subject of a communication to a future meeting of the Johns Hopkins Mathematical Society.

I will here only premise that the seeming contradiction between the logical conclusion and the facts of the case takes its rise in a sort of mirage with which invariantists are familiar, namely: the apparent *à priori* establishment of algebraical forms as the result of perfectly valid processes, which forms have no more real existence in nature than the Corona of the Sun under our Dr Hastings' scrutinizing gaze: the contradiction between the logical inference and the truth being accounted for by the circumstance that any such supposed form on actual performance of the operations indicated, turns out to be a congeries of terms, each affected with a null coefficient; we are thus taught the lesson that all *à priori* reasoning until submitted to the test of experience, is liable to be fallacious, and it is impossible to prove that a proof may not be erroneous by any other method than that of actual trial of the results which it is supposed to yield.

system m and n. Thus, for example, when $\omega = 2$ it is not only true that the determinant of the matrix $mn - nm$ is such a kind of Invariant (which for greater clearness it may be desirable to denote by the term Perpetuitant*), but each element of that matrix will also be a perpetuitant, and these 4 perpetuitants, when for m, n $\rho m, \phi n$ are substituted, will be in an invariable ratio to one another and to either square root of the Involutant.

In like manner it will eventually be seen that for two matrices m, n of any order ω, it is possible to form a matrix of the order ω analogous to $mn - nm$ $\left(\text{which be it observed may be regarded as the Determinant of the matrix } \begin{matrix} m & n \\ m & n \end{matrix}\right)$ each of whose ω^2 terms will be in a constant ratio to each other and to any ωth root of I and of J.

I will now return to the problem of finding what is the form of the equation which connects the ω^2 matrices denoted by

$$(1, m, m^2, \ldots m^{\omega-1})(1, n, n^2, \ldots n^{\omega-1})$$

when such an equation admits of being formed, that is, $I = 0$.

To fix the ideas let us suppose that m, n are matrices of the 3rd order of perfectly general form so that the m, n involution necessitates the satisfaction of one single condition, $I = 0$.

Let $A + Bn + Cn^2 = 0$ be the equation whose form is to be determined where A, B, C, are each of them quadratic functions of m. I say that neither A, B, nor C, can contain a non-vacuous linear factor. For suppose that any one of them as A should contain the non-vacuous factor $m + q$, and that

$$A = (m + q)(am + p).$$

Then we may multiply the equation by $(m + q)^{-1}$ and thus obtain the equation

$$(am + p) + B'n + C'n^2 = 0,$$

that is, we have an equation in which not all 9 but only 8 of the terms signified by $(1, m, m^2)(1, n, n^2) = 0$ are linearly related. But this obviously implies, contrary to the hypothesis, the existence of two equations of condition instead of one.

Hence then A must be of the form $c(m - \lambda)(m - \lambda')$ where λ, λ' are each of them a latent root of m; whether the same or different remains to be determined.

In like manner it may be shown that B is of the form $c_1(m - \lambda_1)(m - \lambda_1')$ and C of the form $c_2(m - \lambda_2)(m - \lambda_2')$. But now I say further that

$$(m - \lambda)(m - \lambda'), \quad (m - \lambda_1)(m - \lambda_1'), \quad (m - \lambda_2)(m - \lambda_2')$$

must be identical.

* *Perpetuitant* formed from *perpetuity* by analogy to Annuitant from Annuity. Perpetuant would have been better, but that it has already been applied by myself in the theory of Invariants in a sense recognized and adopted by Cayley, Hammond, and MacMahon.

For, firstly, suppose that any one pair of the λ's, say λ, λ', are distinct. If any other pair, say λ_2, λ_2', is not identical with this pair, on multiplying the equation by $m-\lambda''$, where λ'' is the 3rd latent root of M, the term containing the term $A(\lambda \ldots \lambda'')$ will vanish, but $B(\lambda \ldots \lambda'')$ will not vanish and consequently there will be an equation, if $C(\lambda \ldots \lambda'')$ does not vanish, between 6 only, and if $C(\lambda \ldots \lambda'')$ does vanish, between 3 only of the 9 terms denoted by $(1, m, m^2)(1, n, n^2)$, contrary to hypothesis.

The only remaining supposition is that A, B, C are each perfect squares. Suppose, then, that any one of them as A is a multiple of $(m-\lambda)^2$; unless B, C are each of them also multiples of the same, on multiplying the equation by $(m-\lambda')(m-\lambda'')$, one of the three coefficients of $1, n, n^2$ will vanish but one at least of the other two will not vanish, which is impossible for the same reason as before. Hence the left-hand side of the equation of involution must contain $(m-\lambda)(m-\lambda')$ as a sinister factor where λ, λ' (whether the same or different) are latent roots of λ. And in like manner precisely, by arranging the equation of involution under the form $A'+mB'+m^2C'$ where A', B', C' are quadratic functions of n, it may be found that the same function must contain $(n-\mu)(n-\mu')$ where μ, μ' are latent roots of n as a dexter factor.

Hence the form of the equation must be

$$(m-\lambda)(m-\lambda')(n-\mu)(n-\mu')=0.$$

It is easy to see that we cannot have λ and λ' the same latent root of m and at the same time μ, μ' the same latent root of n, for then the above product would have at most the nullity 2 whereas it is an absolute null, that is, has the nullity 3.

But I will now show that λ, λ' and μ, μ' must each consist of unlike roots. Let t be any term of the matrix

$$(m-\lambda)(m-\lambda')(n-\mu)(n-\mu'),$$

where t will be a known function of the elements of m, n, of λ, λ' entering symmetrically, and of μ, μ' also entering symmetrically: this is the same thing as saying that t will be a function of the elements of m and n, of λ'', μ'', and of the coefficients of the equations which contain the 3 latent roots of λ and μ respectively.

Consequently the product of the 9 values of t found by writing $\lambda'', \lambda', \lambda$ for λ'', and μ'', μ', μ for μ'', will be a rational integer function of the elements of m, n which vanishes when the Involutant I vanishes and must consequently contain I as a factor. If then, *in any single instance*, the matrix

$$(m-\lambda)^2(n-\mu')(n-\mu'')$$

does not vanish for some one value of λ and μ when I vanishes, it cannot be the form, or *one of two* conceivably possible coexisting forms, of the left-hand side of the general equation of involution. A similar remark of course applies to

$$(m-\lambda_1)(m-\lambda_2)(n-\mu_1)^2.$$

Let now

$$m=\begin{matrix}1 & 0 & 0\\ 0 & \rho & 0\\ 0 & 0 & \rho^2\end{matrix},\qquad n=\begin{matrix}0 & \rho & k\\ k & 0 & \rho^2\\ 1 & k & 0\end{matrix}.$$

The latent roots of m are $1, \rho, \rho^2$, and of n are $\theta, \rho\theta, \rho^2\theta$, where $\theta=\sqrt[3]{(1+k^3)}$; we have also

$$m^2=\begin{matrix}1 & 0 & 0\\ 0 & \rho^2 & 0\\ 0 & 0 & \rho\end{matrix},\qquad n^2=\begin{matrix}-\rho^2 k & k^2 & 1\\ \rho^2 & -k & k^2\\ k^2 & \rho & -\rho k\end{matrix}.$$

The three values of $(m-\lambda')(m-\lambda'')$ are

$$\begin{matrix}3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{matrix},\qquad \begin{matrix}0 & 0 & 0\\ 0 & 3\rho^2 & 0\\ 0 & 0 & 3\rho\end{matrix},\qquad \begin{matrix}0 & 0 & 0\\ 0 & 3\rho & 0\\ 0 & 0 & 3\rho^2\end{matrix},$$

and the three values of $(n-\mu_1)(n-\mu_2)$ are

$$\begin{vmatrix}-\rho^2k+\theta^2 & k^2+\rho\theta & 1+\theta k\\ \rho^2+\theta k & -k+\theta^2 & k^2+\rho^2\theta\\ k^2+\theta & \rho+\theta k & -\rho k+\theta^2\end{vmatrix}\quad \begin{vmatrix}-\rho^2k+\rho^2\theta^2 & k^2+\rho^2\theta & 1+\rho\theta k\\ \rho^2+\rho\theta k & -k+\rho^2\theta^2 & k^2+\theta\\ k^2+\rho\theta & \rho+\rho\theta k & -\rho k+\rho^2\theta^2\end{vmatrix}$$

$$\begin{vmatrix}-\rho^2k+\rho^2\theta & k^2+\theta & 1+\rho^2\theta k\\ \rho^2+\rho^2\theta k & -k+\rho\theta^2 & k^2+\rho\theta\\ k^2+\rho^2\theta & \rho+\rho^2\theta k & -\rho k+\rho\theta^2\end{vmatrix}.$$

The general value of

$$(m-\lambda_1)(m-\lambda_2)(n-\mu_1)(n-\mu_2)$$

will (to a numerical factor *près*) be a matrix consisting of a single column accompanied by two columns of zeros, the non-zero column being some one of the 9 columns found in the above 3 matrices.

Now by direct calculation we know that the n, m Involutant in this case is a numerical multiple of $(k^3-\rho^2)^3$ and vanishes when $k^3=\rho^2$, which gives $\theta=\sqrt[3]{(1+\rho^2)}$, that is, $-\rho=\theta^3$, and if we please $k=\theta^2$.

Hence not merely one but three of the products of

$$(m-\lambda')(m-\lambda'')(n-\mu')(n-\mu'')$$

will in this case vanish, for the above equations will cause the 2nd, 4th and 9th columns all to become columns of nulls.

If now instead of the factor $(m-\lambda')(m-\lambda'')$ we substitute the factor $(m-\lambda)^2$, the three values of $(m-\lambda)^2$ will become

$$\begin{matrix}0 & 0 & 0\\ 0 & -3\rho & 0\\ 0 & 0 & -3\rho^2\end{matrix}\qquad \begin{matrix}-3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -3\rho^2\end{matrix}\qquad \begin{matrix}-3 & 0 & 0\\ 0 & -3\rho & 0\\ 0 & 0 & 0\end{matrix}$$

so that if $$(m-\lambda)^2(n-\mu')(n-\mu'')$$
is to vanish, it will readily be seen that each of two columns of one or the other of the two matrices representing $(n-\mu')(n-\mu'')$ will have to vanish simultaneously, and that this cannot be brought to pass when $\theta^3=-\rho$ and $k^3=\rho^2=\theta^6$ whether we make $k=\theta^2$ or $-\theta^5$ or θ^8.

Hence $$(m-\lambda)^2(n-\mu')(n-\mu'')=0$$
is not an admissible general involution form of equation. Similarly by interchanging the above special values assigned to m and n, it may be shown that
$$(m-\lambda')(m-\lambda'')(n-\mu)^2=0$$
is not an admissible form, and consequently that the one universal form of the involution equation is expressed by saying that
$$(m-\lambda')(m-\lambda'')(n-\mu')(n-\mu'')$$
is an absolute null. If no connexion exists between the elements of m and n, we know from the law of nullity that the above matrix has a nullity 2, that is, that all its minors except the elements themselves have zero contents. The effect of the vanishing of I is to make the elements themselves one and all vanish when the two sets of latent roots are duly selected.

So in general if
$$F=\lambda^{\omega}-A_1\lambda^{\omega-1}+A_2\lambda^{\omega-2}-A_3\lambda^{\omega-3}\ldots=0,$$
and $$G=\mu^{\omega}-B_1\mu^{\omega-1}+B_2\mu^{\omega-2}-B_3\mu^{\omega-3}\ldots=0,$$
are the two equations to the latent roots of m, n matrices of order ω, and if
$$M=m^{\omega-1}-(A_1-\lambda)m^{\omega-2}+(A_2-A_1\lambda+\lambda^2)m^{\omega-3}\ldots$$
and $$N=n^{\omega-1}-(B_1-\mu)n^{\omega-2}+(B_2-B_1\lambda+\lambda^2)n^{\omega-3}\ldots,$$
$MN=0$ for some value of λ and of μ is the one equation of involution, and $NM=0$ for some value of λ and some value of μ is the other such equation.

I will now show how to deduce from the above statement the following marvellous theorem.

Let H represent the sum of the product of each term in the matrix M by its *altruistic opposite* in N (so that H is a function of λ and μ and of degree $\omega-1$ in each of them) then will the ordinary Algebraical Resultant of F, G, H* be exactly equal (in magnitude as well as form) to the product of the two involutants to the *corpus* m, n†.

* The system of equations whose resultant expresses the undifferentiated condition of involution, may be written under the form $(x,y)^{\omega}=0$; $(z,t)^{\omega}=0$; $(x,y)^{\omega-1}=0$. *Quære* whether such a resultant may not be written under the form of a determinant by an application of the Dialytic Method?

† If I and J be the two involutants, $I=0$ will be the condition of left-handed involution of m, n or right-handed of n, m, and $J=0$ of right-handed involution of m, n or left-handed of n, m, for Involution, like light, "has sides." But $IJ=0$ will be the condition of *one* or *the other* kind, or so to say of undifferentiated Involution.

By the theorem proved at the beginning of this note, the nullity of M and that of N are each $\omega - 1$, hence the nullity of MN and consequently *à fortiori* its vacuity cannot be less than $\omega - 1$, and accordingly the identical equation to MN may be written under the form

$$(MN)^{\omega} - H(MN)^{\omega-1} = 0,$$

where H is the sum of the product of each element in the Matrix M or the Matrix N multiplied by its altruistic opposite in the other. Suppose now that $I = 0$ then for some one system of λ, μ out of the ω^2 systems given by the equations $F = 0$, $G = 0$, H must vanish (for the nullity and *à fortiori* the vacuity of MN in that case becomes ω); hence the *double norm* of H, that is, the product of the ω^2 values of H, or, which comes to the same thing, the resultant of F, G, H, must vanish when I vanishes and must therefore contain I; in like manner because the nullity of NM and *à fortiori* its vacuity is ω when $J = 0$, it follows that the same resultant, say R, must contain also J; R will therefore contain IJ, from which it may readily be concluded that it can differ from IJ, if it differ at all, only by a numerical factor.

I need hardly pause to defend the assumption that I, J have no common factor, and that it is the first and not necessarily any higher power of R which contains IJ; the single instance, when

$$m = \begin{matrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{matrix}, \quad n = \begin{matrix} 0 & \rho & k \\ k & 0 & \rho^2 \\ 1 & k & 0 \end{matrix},$$

of I, J being respectively (to a numerical factor *près*) the cubes of $k^3 - \rho$ and $k^3 - \rho^2$ which have no common factor, settles the first part of this assumption at all events for the case of $\omega = 3$, and as regards the second, it is only necessary to show that neither I nor J is equal to, or contains a square or higher power of a function of the letters in m and n as may be done easily enough when $\omega = 3$ by another simple instance*. We may then at once proceed to compare the dimensions of R with those of I and J.

* Limiting ourselves to the case of matrices of the third order, if we take for m, n the matrices $\begin{matrix} 0 & b & 0 \\ d & 0 & f \\ 0 & h & 0 \end{matrix}$, $\begin{matrix} 0 & B & 0 \\ D & 0 & F \\ 0 & H & 0 \end{matrix}$, it may be shown by direct computation that one of the Involutants becomes

$$(bH - hB)^2 (fD - dF)^2 (bd + fh) (BD - FH) (dB - fH) . \{(hF + bD)^2 - (bd + fh) (BD + FH)\},$$

and consequently if there were any square factor in either involutant such factor would contain the elements belonging to the two sets indecomposably blended, but on the other hand, if we take for m, n the matrices $\begin{matrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{matrix}$, $\begin{matrix} 0 & f & F \\ G & 0 & g \\ h & H & 0 \end{matrix}$, either involutant to m, n may easily be shown (also by direct computation) to be made up of three factors, each of which is an indecomposable cubic function of f, g, h, F, G, H. Hence it follows that neither involutant can in its general

R being the product of ω^2 values of $\lambda^{\omega-1}\mu^{\omega-1}+\text{etc.}$, where λ, μ are codimensional with the elements in m and n respectively, is obviously of the degree $\omega^2.(\omega-1)$ in regard to each set of elements, that is, of the degree $2\omega^2(\omega-1)$ in regard to the two sets taken together.

Consider now the degree of I; this is the topical resultant of ω^2 matrices of the form $m^i.n^j$, where

$$i=0,\ 1,\ 2,\ \ldots\ \omega-1,\quad j=0,\ 1,\ 2,\ \ldots\ \omega-1,$$

so that each term in I will consist of a combination of ω^2 elements selected respectively from these ω^2 matrices. If ω is even, there will be $\frac{\omega^2}{2}$ pairs of matrices, one of any such pair of the form m^in^j, the other of form $m^{\omega-1-i}.n^{\omega-1-j}$, and the combination of elements taken from any such pair will be of the collective degree $2(\omega-1)$ in the two sets of elements, so that the total degree of the Involutant will be $\frac{\omega^2}{2}.2(\omega-1)$ or $\omega^2(\omega-1)$. If again ω is odd, there will be $\frac{1}{2}(\omega^2+1)$ such pairs, and one factor (unpaired) belonging to the matrix $m^{\frac{\omega-1}{2}}.n^{\frac{\omega-1}{2}}$ of the collective degree $(\omega-1)$. Hence the degree of the involutant will be

$$(\omega^2-1)(\omega-1)+(\omega-1)\ \text{or}\ \omega^2(\omega-1)$$

as before.

Hence the product of IJ is of the degree $2\omega^2(\omega-1)$, or the same as R, and consequently (at all events to a numerical factor *près*) R and IJ coincide, which is the essential thing to be proved.

N.B. As regards $\omega=3$, the above proof is exact; for higher values of ω to make it valid, it must be demonstrated as a Lemma that the two general twin involutants (even were they decomposable forms, which they undoubtedly are not) could not have any common factor, nor either of them contain any square factor. The Resultant of F, G, H may be compared to a cradle just large enough to contain the twin forms in question, so as to give assurance that no other form is mixed up with them; and the proof given above shows that this must be the case if neither twin is doubled

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form contain any square factor. As a matter of fact, not only for ternary matrices but for matrices of any order, there can be no reasonable doubt whatever in any sane mind that every Involutant is *absolutely* indecomposable. One must try, however, to obtain a strict proof of this upon the general principle of crushing every logical difficulty regarded as a challenge to the human reason, which falls in our way; it is in overcoming the difficulties attendant upon the proof of negative propositions that the mind acquires new strength and accumulates the materials for future and more significant conquests. To prove that involutants in their general form are indecomposable may possibly, I imagine, prove to be a hard nut to crack, or it may be exceedingly easy.

up upon itself, and if the two do not grow into one another, but like such creatures each possesses a perfectly distinct organization.

A single instance will serve to establish the fact that the Resultant of F, G, H is the very product IJ itself, without any numerical multiplier. I have made this verification for binary and ternary matrices, and as the point is not one of an essential importance need not dwell here further upon it.

To pass to a much more important subject, I am inclined to anticipate as the result of a long and interesting investigation into the relations of the involutants of a certain particular *corpus* of the third order that the *sum* of the two involutants of any *corpus* admits of being represented by means of invariants similar in kind to that which expresses the single involutant to a binary *corpus* (m, n), namely, the content of (that is, the determinant to) the matrix $mn - nm$, which itself (as previously observed) may be written as the determinant to the matrix $\begin{Bmatrix} m & n \\ m & n \end{Bmatrix}$, or say $(m, n)_2$; and in some similar way it is, I think, not unlikely that the *product* also of the two involutants (the resultant of F, G, H) is capable of being expressed; but I must for the present content myself with exhibiting the bare fact of the existence of invariants of the kind referred to for matrices of any order.

Suppose then that m, n is a *corpus* of the third order. Form the determinant

$$\left.\begin{matrix} m & n & m^2 & n^2 \\ m & n & m^2 & n^2 \\ m & n & m^2 & n^2 \\ m & n & m^2 & n^2 \end{matrix}\right\}, \text{ say } (m, n, m^2, n^2)_4.$$

The number of terms, half of them positive and half of them negative, in such determinant is 24; but of these, all but 8 will obviously appear as pairs of equal terms affected with opposite signs and so cancel one another: the 8 excepted ones are those in which no m and n come together, *to wit*:

$$mnm^2n^2 + nmn^2m^2 + m^2n^2mn + n^2m^2nm$$
$$- m^2nmn^2 - nm^2n^2m - mn^2m^2n - n^2mnm^2.$$

The determinant to this matrix will be of the total degree 18 in the two sets of elements belonging to m and n respectively, that is, of the degree 9 in respect to each set of elements *per se*. And so in general if m, n be of the order ω the determinant

$$(m, m^2, \ldots m^{\omega-1}, n, n^2, \ldots n^{\omega-1})_{2\omega}$$

will contain only $2(\pi\omega)^2$ effective terms, of which half will bear the positive and the others the negative sign.

The determinant to this matrix will be of the order

$$\omega [2 \{1 + 2 + \dots + (\omega - 1)\}], \text{ that is, } (\omega - 1) \omega^2,$$

in regard to the combined elements in m and n, that is, equi-dimensional with either involutant to the *corpus* m, n.

Whatever else may be its properties (on which I do not dare yet to pronounce), it is certain that such determinant (and over and above that, every term in the matrix of which it is the content) will be an Invariant to the *corpus* in the same sense in which either Involutant has been previously shown to be entitled to bear that name. And here for the present it becomes necessary for me to break off, bidding *au revoir* to any reader who may peruse this sketch, and trusting to meet him again in the broader field of the *American Journal of Mathematics*, where I hope to be spared to set out this portion of the theory with more certainty, and the whole doctrine of multiple quantity with much greater completeness and in more ample detail than is possible within the limits of the *Circulars* and in the short interval remaining between the present time and the date of my intended departure for Europe.