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BRIEF NOTES

Brittle fracture as a wave

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THE PURPOSE of this note is to show that a simple self-consistent theory of crack propagation may be obtained if crack propagation is considered as a dynamic process. The basic assumption is: Each elementary crack produces a wave. In order to expose the idea the calculations are performed for a very simple case.

1. Elementary wave

CONSIDER a plane crack propagating with constant speed p in linear elastic material. At time t the crack occupies the half-plane $x < pt$, $y = 0$ (Fig. 1). Assume that the displacement vector has the components of the form

(1.1) $$u^1 = u^2 = 0, \quad u^3 = u(x, y, t).$$

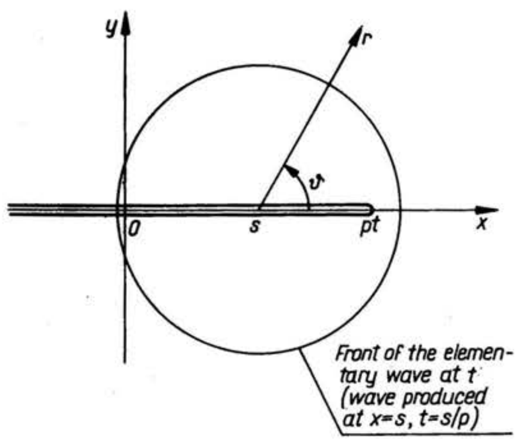

FIG. 1.

Elementary fracture at $x = s$ of the length ds is the source of a wave propagating in the direction perpendicular to the z-axis. This wave will be called the elementary wave. Since it is a shear wave its speed is

(1.2) $$U = \sqrt{\mu/\varrho},$$

where ϱ is the material density and μ the shear modulus. The elementary wave produced at $x = s$ starts at time $t = s/p$.

Introduce the cylindrical coordinates (r, ϑ, z) with the origin at $x = s$, $y = z = 0$, and denote

$$\Phi = \frac{r}{U} - \left(t - \frac{s}{p}\right). \tag{1.3}$$

At the front of the elementary wave there is $\Phi = 0$. The derivatives of Φ with respect to r, ϑ, z are the components of the wave vector

$$\left(\frac{1}{U}, 0, 0\right). \tag{1.4}$$

Represent the displacement du produced by the elementary wave in the form of the series

$$du(r, \vartheta, z, t) = \sum_{\nu=0}^{\infty} S_{\nu+2}(\Phi) g_\nu(r, \vartheta, z, t), \tag{1.5}$$

where the functions S_ν satisfy the recursive formula

$$\frac{dS_\nu(\Phi)}{d\Phi} = S_{\nu-1}(\Phi), \tag{1.6}$$

and the coefficients $g_\nu(r, \vartheta, z, t)$, $\nu = 1, 2, 3, \dots$ are unknown functions. After substituting Eq. (1.5) into the Lame equations the infinite set of equations for g_ν may be obtained.

If it is assumed that

$$S_2(\Phi) = \Phi^2, \tag{1.7}$$

then the elementary calculations exposed, for example, in [1] lead to the solution

$$du = \begin{cases} \dfrac{1}{2} B \dfrac{ds}{\sqrt{r}} \left(\dfrac{r}{U} - t + \dfrac{s}{p}\right)^2 \sin\dfrac{1}{2}\vartheta & \text{for} \quad r < U\left(t - \dfrac{s}{p}\right), \\ 0 & \text{for} \quad r > U\left(t - \dfrac{s}{p}\right), \end{cases} \tag{1.8}$$

where B represents the intensity of the elementary wave. Time derivatives of the expression (1.8) satisfy the Lamé equations, too. Take in particular the second time derivative, namely

$$d\overset{*}{u} = \begin{cases} B \dfrac{ds}{\sqrt{r}} \sin\dfrac{1}{2}\vartheta & \text{for} \quad r < U\left(t - \dfrac{s}{p}\right), \\ 0 & \text{for} \quad r > U\left(t - \dfrac{s}{p}\right). \end{cases} \tag{1.9}$$

Because of the relations

$$r = \sqrt{(x-s)^2 + y^2}, \quad \cos\vartheta = \frac{x-s}{r}, \tag{1.10}$$

we have

$$d\overset{*}{u} = \begin{cases} B \dfrac{ds}{\sqrt{2}} \dfrac{\sqrt{\sqrt{(x-s)^2 + y^2} - (x-s)}}{\sqrt{(x-s)^2 + y^2}} & \text{for} \quad r < U\left(t - \dfrac{s}{p}\right), \\ 0 & \text{for} \quad r > U\left(t - \dfrac{s}{p}\right). \end{cases} \tag{1.11}$$

2. Total displacement

All the elementary waves add together and produce the total displacement u

(2.1) $$u(x, y, z, t) = \int_{s=-\infty}^{s=pt} d\overset{*}{u}(x, y, z, t, s).$$

In order to perform the integration note that in accord with Eq. (1.11), some of the elementary waves can not contribute to the total displacement u. The latest elementary wave that does contribute is produced at $x = s_0$, where (cf. Fig. 2)

(2.2) $$r_0 = U\left(t - \frac{s_0}{p}\right), \quad r_0 = \sqrt{(x-s_0)^2+y^2}\,.$$

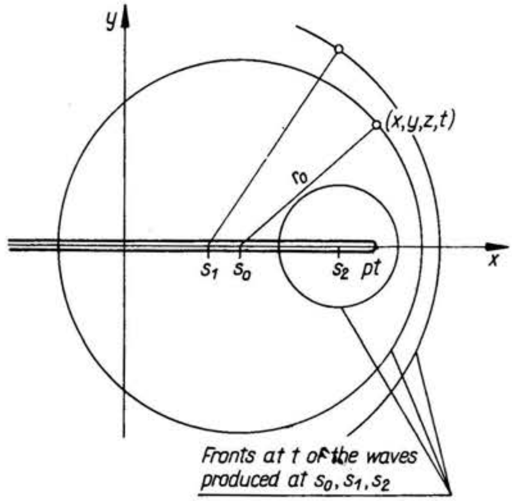

FIG. 2.

After solving the above equation we obtain

(2.3) $$s_0 = -\frac{p^2}{U^2-p^2}\left[x - \frac{U^2}{p}t + \frac{U}{p}\sqrt{(x-pt)^2+\left(1-\frac{p^2}{U^2}\right)y^2}\,\right].$$

The expression (2.1) reduces now to the integral

(2.4) $$u = \frac{B}{\sqrt{2}}\int_{s=-\infty}^{s_0} \frac{\sqrt{\sqrt{(x-s)^2+y^2}-(x-s)}}{\sqrt{(x-s)^2+y^2}}\,ds.$$

Since all the waves produced at $s < s_0$ contribute to u, the expression valid for $r < U\left(t - \frac{s}{p}\right)$ was taken. The integral of Eq. (2.4) is

$$2\sqrt{\sqrt{(x-s)^2+y^2}-(x-s)}\,,$$

therefore,

(2.5) $$u = \pm B\sqrt{2}\sqrt{\sqrt{(x-s_0)^2+y^2}-(x-s_0)}\,.$$

Substitute now Eq. (2.3) into Eq. (2.5) to obtain the final result

$$u = \pm \frac{B\sqrt{2}}{\sqrt{1+p/U}} \sqrt{\sqrt{(x-pt)^2+(1-p^2/U^2)y^2}-(x-pt)}\,. \tag{2.6}$$

The "+" sign has to be taken for $y > 0$, and the "−" sign has to be taken for $y < 0$.

Denote

$$C = \frac{B}{\sqrt{1+p/U}}\,.$$

From Eq. (2.6) we obtain the following formulae:

$$u = \begin{cases} 2C\sqrt{|x-pt|}\,, \\ 0 \end{cases} \quad \frac{\partial u}{\partial y} = \begin{cases} 0 & \text{for} \quad x < pt, \quad y = 0, \\ C\sqrt{\dfrac{1-p^2/U^2}{x-pt}} & \text{for} \quad x > pt, \quad y = 0. \end{cases} \tag{2.7}$$

The stress vector on the crack equals $\pm\mu\dfrac{\partial u}{\partial y}$. It is seen that on the crack the stress vector equals zero.

We shall now show that Eq. (2.6) represents the surface wave. Define

$$\tilde{u} = im\sqrt{\frac{x}{p}+i\frac{y}{q}-t}\,, \tag{2.8}$$

where

$$\frac{1}{q^2} = \frac{1}{U^2}-\frac{1}{p^2}. \tag{2.9}$$

Elementary calculations lead to the formula

$$\tilde{u} = \sqrt{\sqrt{(x-pt)^2+(1-p^2/U^2)y^2}-(x-pt)}. \tag{2.10}$$

It follows that $\tilde{u}$ is proportional to u, as given by Eq. (2.6).

On the other hand the expression (2.8) has the form of the expansion (1.5) with

$$\Phi^* = \frac{x}{p}+i\frac{y}{q}-t, \tag{2.11}$$

$$S_2^*(\Phi^*) = \sqrt{\Phi^*}. \tag{2.12}$$

The asterisk has been added to make a distinction between Eqs. (1.3), (1.7) and the above formulae. The corresponding wave vector is complex and has the components

$$\frac{1}{p}, \quad i\frac{1}{q}, \quad 0.$$

Therefore, Eq. (2.10) is a surface wave. The front of this wave as given by the equation $\Phi^* = 0$ is not real. One point of this front, namely the point $x = pt$ is real, and its speed equals p.

3. Propagation of the crack

Consider the strip shown in Fig. 3 and calculate the work L done by the external forces in time δt. The work done at $y = +h$ equals that done at $y = -h$. The stress

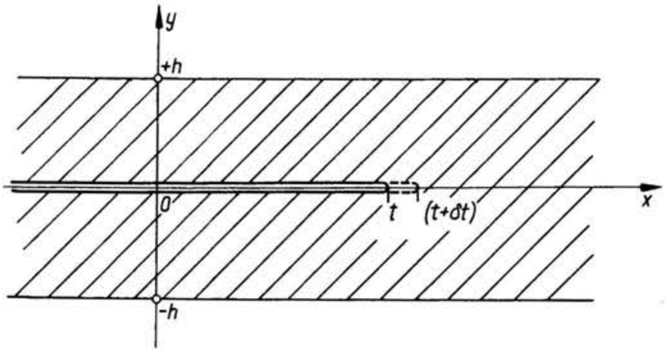

FIG. 3.

vector at $y = +h$ equals $\mu \, \partial u / \partial y|_{y=h}$ and the additional displacement equals $\partial u / \partial t \, \delta t$. Therefore,

$$L = 2\mu \, \delta t \int_{-\infty}^{\infty} \left(\frac{\partial u}{\partial y} \frac{\partial u}{\partial t} \right) \Bigg|_{y=h} dx. \tag{3.1}$$

After substituting into this relation the expression (2.6), we obtain the integral

$$L = C^2 \mu h \left(1 - \frac{p^2}{U^2} \right) p \delta t \int_{-\infty}^{\infty} \frac{dx}{(x-pt)^2 + A^2 h^2}. \tag{3.2}$$

The final result is

$$L = \pi \dot{p} \delta t C^2 \mu \sqrt{1 - p^2/U^2}. \tag{3.3}$$

Note that L is independent of h. In the limit case $h \to 0$ we obtain the same L. It follows that L is totally used for producing the crack of length $p\delta t$. If Q denotes the energy necessary to produce a unit area of crack (cf. e.g. [2] or [3]) in accord with Eq. (3.3) the equality

$$Qp\delta t = \pi p \delta t C^2 \mu \sqrt{1 - p^2/U^2}$$

must hold.

By dividing by $p\delta t$, we finally obtain

$$p^2 = U^2 \left(1 - \frac{Q^2}{\pi^2 \mu^2 C^4} \right). \tag{3.4}$$

The method shown above allows to produce similar results for other, more complex cases of brittle fracture. In particular, the fracture with non-constant speed or not-plane crack may be considered by the same method.

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