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Singular solutions in microcontinuum fluid mechanics

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THE METHOD of associated matrices is applied to obtain Galerkin-type representations for the equations of motion of two basic linear theories in microcontinuum fluid mechanics. The representations are then utilized in constructing singular solutions for these theories. The results, wherever possible, are compared and are found to be in agreement with those of previous investigators.

Zastosowano metodę macierzy stowarzyszonych do wyprowadzenia reprezentacji typu Galerkina dla równań ruchu dwóch podstawowych teorii liniowych mikrokontynualnej mechaniki cieczy. Reprezentacje te wykorzystano następnie do skonstruowania rozwiązań osobliwych dla tych teorii. Wyniki zostały, w miarę możności, porównane z wynikami uzyskanymi przez innych badaczy.

Применен метод присоединенных матриц для вывода представлений типа Галеркина для уравнений движения двух основных линейных теорий микроконтинуальной механики жидкостей. Эти представления использованы затем для построения особых решений для этих теорий. Результаты сравнены, по мере возможности, с результатами полученными другими исследователями.

Notations

d^2 Laplace operator,
$\bar{f}$ body force,
$\bar{F}$ constant body force,
I unit matrix,
$\bar{l}$ body couple,
$\bar{L}$ constant body couple,
p dynamic pressure field,
$\bar{r}$ position vector of an arbitrary point x with respect to an origin at y,
r distance of x from origin y,
$\bar{u}$ velocity vector,
X, Y, Z matrices $(2 \cdot 1)$,
$\delta(x-y)$ Dirac delta function,
$(\gamma, \mu, \eta, \alpha, \theta, \tau)$ material constants,
λ^2 $\dfrac{2\gamma\mu}{\theta(\gamma-\mu)}$,
σ^2 $\dfrac{-2\gamma}{(\eta+\tau+\theta)}$,
$\bar{\Omega}$ "inherent" angular velocity,
∇ gradient operator,
∇^2 Laplace operator.

Subscripts

k integer running over values 0, 1, 2, 3,
i, j, m, n integers running over values 1, 2, 3.

1. Introduction

THE CONSIDERATION of couple-stress in addition to the classical Cauchy stress, has led to the recent development of theories of fluid microcontinua. This new branch of fluid mechanics has attracted a growing interest during recent years mainly because it possesses the mechanism to describe such rheologically complex fluids as liquid crystals, polymeric suspensions and animal blood.

The many and diverse applications of microcontinuum mechanics to flow problems are well documented in a review article by ARIMAN et al. [1]. KLINE and ALLEN [2] studied blood flow based on a microcontinuum formulation by investigating the concentration effects in oscillatory blood flow. More recently with an improved microcontinuum model of blood, ARIMAN et al. [3] presented an encouraging comparison of the theoretical velocity and cell-rotational velocity profiles with the experimental data of BUGLIARELLO and SEVILLA [4].

In view of these findings it seems logical that further research work, both theoretical and experimental, will be undertaken in this area. With this in mind, the following work is presented as foundation material for possible use in future studies.

This paper is concerned with finding the singular solutions for the slow steady field equations of two apparently different specialized theories of microcontinuum fluids — Stokes' couple stress theory [5] which contains first and second order velocity gradients and the theory of asymmetric hydromechanics developed by AERO et al. [6] in which a new independent kinematic variable, the gyration tensor, is introduced. The method of associated matrices which has been used by several authors [7, 8] in the study of elasticity and linear elastic dielectrics, is employed to construct Galerkin-type representations. With the aid of these representations, the required singular solutions are obtained.

2. Basic results

Let

$$(2.1)\qquad I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\qquad X=\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix},$$
$$Y=\begin{bmatrix}0&-X_3&X_2\\X_3&0&-X_1\\-X_2&X_1&0\end{bmatrix},\quad Z=\begin{bmatrix}X_1^2&X_1X_2&X_1X_3\\X_2X_1&X_2^2&X_2X_3\\X_3X_1&X_3X_2&X_3^2\end{bmatrix}$$

be matrices with elements as real numbers. If $d^2 = X_1^2+X_2^2+X_3^2$ and the superscript "t" over a matrix denotes its transpose, then the following relations are easy to verify:

$$(2.2)\qquad \begin{aligned} &X^tX = d^2, \quad X^tY = 0, \quad X^tZ = d^2X^t,\\ &YX = 0, \quad Y^2 = -d^2I+Z, \quad YZ = 0,\\ &ZX = d^2X, \quad ZY = 0, \quad Z^2 = d^2Z,\\ &XX^t = Z. \end{aligned}$$

Working in Cartesian coordinates (x_1, x_2, x_3), let $\frac{\partial}{\partial x_i} = X_i$ $(i = 1, 2, 3)$; it then follows that

$$\nabla^2 = d^2, \quad \bar{\nabla}\times\bar{V} = Y\bar{V}, \quad \bar{\nabla}\bar{\nabla}\cdot\bar{V} = Z\bar{V},$$

where $\bar{V}$ is a vector represented by the column matrix $[V_1, V_2, V_3]^t$.

In order to construct the three-dimensional singular solutions in subsequent sections, we note that if

$$(2.3) \quad [\nabla^2, \nabla^4, \nabla^2+a_0^2, (\nabla^2+a_1^2)(\nabla^2+a_2^2), (\nabla^2+a_1^2)(\nabla^2+a_2^2)(\nabla^2+a_3^2)]g = -\delta(x-y)[1, 1, 1, 1, 1],$$

then the corresponding solutions are given by

$$(2.4) \qquad g = \frac{1}{4\pi}\left[\frac{1}{r}, \frac{r}{2}, \frac{e^{ia_0r}}{r}, \frac{e^{ia_1r}-e^{ia_2r}}{r(a_1^2-a_2^2)}, \sum_{j=1}^{3} E_j\frac{e^{ia_jr}}{r}\right]$$

where $\delta(x-y)$ is the Dirac delta function, a_k $(k = 0, 1, 2, 3)$ are arbitrary real constants, $x = (x_1, x_2, x_3)$, $r^2 = \sum_{i=1}^{3}(x_i-y_i)^2$ and $E_j = [(a_j^2-a_m^2)(a_j^2-a_n^2)]^{-1}$, $j \neq m \neq n, j, m, n = 1, 2, 3$.

The matrix inversion technique shall now be illustrated by applying it to the Navier-Stokes equations and moreover we need the results for future reference. In the case of slow steady incompressible flow, an assumption that will be made throughout this work, the equations take the form

$$(2.5) \qquad \begin{aligned} \mu\nabla^2\bar{u}-\bar{\nabla}p &= -\bar{f}, \\ \bar{\nabla}\cdot\bar{u} &= 0. \end{aligned}$$

This system (2.5) is equivalent to the matrix equation

$$(2.6) \qquad A\begin{bmatrix}\bar{u}\\ p\end{bmatrix} = \begin{bmatrix}-\bar{f}\\ 0\end{bmatrix},$$

where the matrix A is given by

$$(2.7) \qquad A = \begin{bmatrix}\mu d^2 I & -X\\ X^t & 0\end{bmatrix}.$$

The solution of (2.6) is of the form

$$(2.8) \qquad \begin{bmatrix}\bar{u}\\ p\end{bmatrix} = A^{-1}\begin{bmatrix}-\bar{f}\\ 0\end{bmatrix},$$

and so the problem reduces to that of finding the inverse matrix A^{-1} of (2.7). This is found to be

$$(2.9) \qquad A^{-1} = \begin{bmatrix}\dfrac{d^2I-Z}{\mu d^4} & \dfrac{X}{d^2}\\ -\dfrac{X^t}{d^2} & \mu\end{bmatrix}.$$

Substitution of (2.9) into (2.8) produces the following Galerkin-type representations for $\bar{u}, p$:

$$\bar{u} = \nabla^2\bar{\phi} - \bar{\nabla}\bar{\nabla}\cdot\bar{\phi} + \bar{\nabla}\psi, \quad p = -\mu\nabla^2\bar{\nabla}\cdot\bar{\phi}, \tag{2.10}$$

where $\bar{\phi}$, ψ satisfy the equations

$$\mu\nabla^4\bar{\phi} = -\bar{f}, \quad \nabla^2\psi = 0. \tag{2.11}$$

Hence

$$\nabla^2 p = \bar{\nabla}\cdot\bar{f}. \tag{2.12}$$

To determine $\bar{u}$, p we need $\bar{\phi}$, ψ; hence we must solve the system (2.11) assuming that the body force f is known. In the case of an arbitrary constant concentrated point force $\bar{f} = \bar{F}\delta(x-y)$ applied at the point $y(y_1, y_2, y_3)$, the solution of (2.11) is given by

$$\bar{\phi} = \frac{1}{8\pi\mu}\bar{F}r, \quad \psi = 0. \tag{2.13}$$

Substituting this into (2.10), we obtain

$$\bar{u} = \frac{1}{8\pi\mu}\left[\frac{\bar{F}}{r} + \frac{\bar{F}\cdot\bar{r}}{r^3}\bar{r}\right], \quad p = -\frac{1}{4\pi}\bar{F}\cdot\bar{\nabla}\left(\frac{1}{r}\right) = \frac{1}{4\pi}\frac{\bar{F}\cdot\bar{r}}{r^3}, \tag{2.14}$$

where the gradient is evaluated at the point y. LAMB [9] gives similar expressions for a point force in the creeping motion approximation using an alternative approach.

3. Stokes' couple stress theory

Developed by STOKES [5] in 1966, this theory represents the simplest generalization of the classical theory which allows for polar effects such as the presence of couple stresses and body couples. But unlike ERINGEN's micropolar fluid theory [10] and the asymmetric hydromechanics of AERO et al. [6], this couple stress theory of fluids defines the rotation field in terms of the velocity field. In fact, the rotation vector is equal to one-half the curl of the velocity vector as is the case in Newtonian fluids. Second order gradient of the velocity vector, rather than the kinematically independent rotation vector of asymmetric hydromechanics, is introduced into the stress constitutive equations and consequently the theory yields only one vector equation to describe the velocity field. In the case of incompressible creeping flow, the field equations are [5]:

$$\mu\nabla^2\bar{u} - \alpha\nabla^4\bar{u} - \bar{\nabla}p = -\bar{b}, \quad \bar{\nabla}\cdot\bar{u} = 0, \tag{3.1}$$

where $\bar{b} = \bar{f} + {}^1/_2\bar{\nabla}\times\bar{l}$.

To obtain the singular solutions, we once again write (3.1) in the matrix form (2.6), i.e

$$A\begin{bmatrix}\bar{u}\\ p\end{bmatrix} = \begin{bmatrix}-\bar{b}\\ 0\end{bmatrix}, \tag{3.2}$$

where

$$A = \begin{bmatrix}(\mu d^2-\alpha d^4)I & -X\\ X^t & 0\end{bmatrix}.$$

The inverse matrix A^{-1} is found to be

$$A^{-1} = \begin{bmatrix}\dfrac{I}{\mu d^2-\alpha d^4}-\dfrac{Z}{d^2(\mu d^2-\alpha d^4)} & \dfrac{X}{d^2}\\ -\dfrac{X^t}{d^2} & \mu-\alpha d^2\end{bmatrix}. \tag{3.3}$$

Substitution of (3.3) into (3.2) leads to the following Galerkin-type representations:

$$\begin{aligned}\bar{u} &= -(\bar{\nabla}\times\bar{\nabla}\times\bar{\phi})+\bar{\nabla}\psi,\\ p &= -(\mu\nabla^2-\alpha\nabla^4)\bar{\nabla}\cdot\bar{\phi}+\nabla^2(\mu-\alpha\nabla^2)\psi,\end{aligned} \tag{3.4}$$

where $\bar{\phi}$, ψ satisfy the equations

$$\begin{aligned}\nabla^2(\mu\nabla^2-\alpha\nabla^4)\bar{\phi} &= -\bar{b},\\ \nabla^2\psi &= 0.\end{aligned} \tag{3.5}$$

If we take $\bar{b} = \bar{F}\delta(x-y)$ where $\bar{F}$ is once more an arbitrary constant vector, then the solution of (3.5) is given by

$$\begin{aligned}\bar{\phi} &= \frac{\bar{F}r}{8\pi\mu}-\alpha\bar{F}\frac{(1-e^{-\sqrt{\frac{\mu}{\alpha}}r})}{4\pi\mu^2 r},\\ \psi &= 0.\end{aligned} \tag{3.6}$$

Using (3.6) and (3.4), the singular solution of (3.1) is finally given by

$$\begin{aligned}\bar{u} &= \frac{1}{8\pi\mu}\left[\frac{\bar{F}}{r}+\frac{(\bar{F}\cdot\bar{r})}{r^3}\bar{r}\right]+\frac{\alpha}{4\pi\mu^2}\bar{\nabla}\times\bar{\nabla}\times\bar{F}\frac{1-e^{-\sqrt{\frac{\mu}{\alpha}}r}}{r},\\ p &= \frac{1}{4\pi}\frac{\bar{F}\cdot\bar{r}}{r^3}.\end{aligned} \tag{3.7}$$

We now make the following observations:

(a) The pressure field is identical to that of the classical theory given by $(2.14)_2$.

(b) The velocity vector $\bar{u}$ is decomposed in the form

$$\bar{u} = \bar{u}_c+\bar{u}_s,$$

where $\bar{u}_c$ is the classical solution $(2.14)_1$ and $\bar{u}_s$ is the contribution due to the couple stresses. As $\alpha \to 0$, $\bar{u} \to \bar{u}_c$ as expected.

(c) BLEUSTEIN and GREEN [11] presented a theory of dipolar fluids with similar field equations in the linearized case. Hence the solutions obtained here will also be valid for such fluids.

4. Asymmetric hydromechanics

By introducing the velocity field $\bar{u}$, an "inherent" angular velocity field $\bar{\Omega}$ and a dissipation function, AERO et al. [6] constructed the equations of a fluid characterized by asymmetric hydromechanics with the aid of the rheological laws. The key point to note in the development of this theory is the introduction of two basic and independent kinematical vector fields — the vector field representing the velocities of the fluid particles and that representing the angular velocities of the fluid particles. The basic field equations are

$$(\mu-\gamma)\nabla^2\bar{u}-2\gamma\bar{\nabla}\times\bar{\Omega}-\bar{\nabla}p+\bar{f}=0,$$
$$(\eta+\tau+\theta)\bar{\nabla}\,\bar{\nabla}\cdot\bar{\Omega}-\theta\bar{\nabla}\times\bar{\nabla}\times\bar{\Omega}+2\gamma\bar{\Omega}-\gamma\bar{\nabla}\times\bar{u}+\bar{l}=0, \tag{4.1}$$
$$\bar{\nabla}\cdot\bar{u}=0.$$

The coefficient γ links together the velocity and the angular velocity fields and may be termed by the coupling constant since its vanishing uncouples the differential equations and produces the classical Navier-Stokes equation. The matrix form of the system of equations (4.1) is

$$A\begin{bmatrix}\bar{u}\\ \bar{\Omega}\\ p\end{bmatrix}=\begin{bmatrix}-\bar{f}\\ -\bar{l}\\ 0\end{bmatrix}, \tag{4.2}$$

where the matrix A is given by

$$A=\begin{bmatrix}L_1 I & -2\gamma Y & -X\\ -\gamma Y & L_2 I+(\eta+\tau)Z & 0\\ X^t & 0 & 0\end{bmatrix}, \tag{4.3}$$

and where $L_1=(\mu-\gamma)d^2$, $L_2=(\theta d^2+2\gamma)$.

After some working, the inverse matrix A^{-1} is obtained in the form

$$A^{-1}=\begin{bmatrix}\dfrac{L_2 d^2 I-L_2 Z}{L_3 d^2} & \dfrac{2\gamma Y}{L_3} & \dfrac{X}{d^2}\\[2ex] \dfrac{\gamma Y}{L_3} & \dfrac{L_1 L_4 I+\{2\gamma^2-(\eta+\tau)L_1\}Z}{L_3 L_4} & 0\\[2ex] \dfrac{-X^t}{d^2} & 0 & (\mu-\gamma)\end{bmatrix} \tag{4.4}$$

so that with the aid of (4.2) we obtain

$$\begin{bmatrix}\bar{u}\\ \bar{\Omega}\\ p\end{bmatrix}=\begin{bmatrix}\dfrac{L_2 d^2 I-L_2 Z}{L_3 d^2} & \dfrac{2\gamma Y}{L_3} & \dfrac{X}{d^2}\\[2ex] \dfrac{\gamma Y}{L_3} & \dfrac{L_1 L_4 I+\{2\gamma^2-(\eta+\tau)L_1\}Z}{L_3 L_4} & 0\\[2ex] \dfrac{-X^t}{d^2} & 0 & (\mu-\gamma)\end{bmatrix}\begin{bmatrix}-\bar{f}\\ -\bar{l}\\ 0\end{bmatrix}, \tag{4.5}$$

where $L_3=L_1L_2+2\gamma d^2$ and $L_4=L_2+(\eta+\tau)d^2$.

The representations for $\bar{u}$, $\bar{\Omega}$ and p are now given by

$$\bar{u} = (\theta\nabla^2+2\gamma)\nabla^2\bar{\phi}_1-(\theta\nabla^2+2\gamma)\bar{\nabla}\bar{\nabla}\cdot\bar{\phi}_1+2\gamma[(\eta+\tau+\theta)\nabla^2+2\gamma]\bar{\nabla}\times\bar{\phi}_2+\bar{\nabla}\Psi,$$

(4.6) $$\bar{\Omega} = \gamma\nabla^2(\bar{\nabla}\times\bar{\phi}_1)+(\mu-\gamma)\nabla^2[(\eta+\tau+\theta)\nabla^2+2\gamma]\bar{\phi}_2 +[2\gamma^2-(\eta+\tau)(\mu-\gamma)\nabla^2]\bar{\nabla}\bar{\nabla}\cdot\bar{\phi}_2,$$

$$p = -\nabla^2[\theta(\mu-\gamma)\nabla^2+2\mu\gamma]\bar{\nabla}\cdot\bar{\phi}_1,$$

where $\bar{\phi}_1$, $\bar{\phi}_2$, ψ satisfy the equations

$$\nabla^4[\theta(\mu-\gamma)\nabla^2+2\gamma\mu]\bar{\phi}_1 = -\bar{f},$$

(4.7) $$\nabla^2[(\eta+\tau+\theta)\nabla^2+2\gamma][\theta(\mu-\gamma)\nabla^2+2\gamma\mu]\bar{\phi}_2 = -\bar{l},$$

$$\nabla^2\Psi = 0.$$

Again, as in the classical case (2.12), $\nabla^2 p = \bar{\nabla}\cdot\bar{f}$. The theory of asymmetric hydromechanic gives rise to two independent fundamental singular solutions of the field equations; namely the velocity and rotational fields in an unbounded medium due to a concentrated body force as well as a concentrated body couple.

(a) Concentrated force

Let $\bar{f} = \bar{F}\delta(x-y)$, $\bar{l} = 0$.

Using (2.3), the solution of (4.7) is found to be

(4.8) $$\bar{\phi}_1 = \frac{\bar{F}}{8\pi\lambda^2}\left[\frac{r}{\theta(\gamma-\mu)}-\frac{(1-e-\lambda r)}{\mu\gamma r}\right],$$

$$\bar{\phi}_2 = 0, \quad \Psi = 0,$$

where

$$\lambda^2 = \frac{2\gamma\mu}{\theta(\gamma-\mu)}.$$

Substitution of (4.8) into (4.6) gives

$$\bar{u} = \frac{1}{4\pi\lambda^2(\gamma-\mu)}\left[\frac{\gamma}{\theta}\frac{\bar{F}}{r}+\frac{\gamma}{\theta}\frac{(\bar{F}\cdot\bar{r})}{r^3}\bar{r}-\frac{(2\gamma-\theta\lambda^2)}{\theta\lambda^2}\left\{\bar{\nabla}\times\bar{\nabla}\times\bar{F}\frac{(1-e^{-\lambda r})}{r}\right\}\right],$$

(4.9) $$\bar{\Omega} = \frac{1}{8\pi\mu}\bar{F}\times\bar{\nabla}\frac{1-e^{-\lambda r}}{\mathrm{r}},$$

$$p = \frac{1}{4\pi}\frac{\bar{F}\cdot\bar{r}}{r^3}.$$

Except for minor changes in viscosity coefficients, the results (4.9) agree with those given in [12] for micropolar fluids whose field equations are similar for creeping flow. It is of interest to note that comparison with (3.7) shows that the velocity and pressure fields are identical in form to those in Stokes' theory.

(b) Concentrated couple

Here we take $\bar{f} = 0$, $\bar{l} = \bar{L}\delta(x-y)$.
In this case the solution of the system (4.7) is given by

$$\bar{\phi}_1 = 0,$$

$$\text{(4.10)}\qquad \bar{\phi}_2 = \frac{\bar{L}}{4\pi\theta(\eta+\tau+\theta)(\mu-\gamma)}\left[\frac{1}{\sigma^2\lambda^2 r} + \frac{e^{-\sigma r}}{\sigma^2(\sigma^2-\lambda^2)r} + \frac{e^{-\lambda r}}{\lambda^2(\lambda^2-\sigma^2)r}\right],$$

$$\Psi = 0,$$

where $\sigma^2 = \dfrac{-2\gamma}{\eta+\tau+\theta}$.

Substituting (4.10) into (4.6), we obtain the following singular solutions for a concentrated couple:

$$\bar{u} = \frac{1}{8\pi\mu}\bar{\nabla}\times\bar{L}\frac{1-e^{-\lambda r}}{r},$$

$$\text{(4.11)}\qquad \bar{\Omega} = \frac{\bar{L}e^{-\lambda r}}{4\pi\theta r} + \frac{\bar{\nabla}\,\bar{\nabla}\cdot\bar{L}}{4\pi\theta(\mu-\gamma)(\eta+\tau+\theta)}\left[\frac{2\gamma^2}{\sigma^2\lambda^2 r} + \frac{2\gamma^2+\lambda^2(\eta+\tau)(\mu-\gamma)}{\lambda^2(\lambda^2-\sigma^2)}\frac{e^{-\lambda r}}{r} + \frac{2\gamma^2+\sigma^2(\eta+\tau)(\mu-\gamma)}{\sigma^2(\sigma^2-\lambda^2)}\frac{e^{-\sigma r}}{r}\right],$$

$$p = 0.$$

5. Conclusions

Fundamental singular solutions of two basic and apparently different linear theories of microcontinuum fluid mechanics are obtained with the aid of a matrix inversion technique. Except for minor changes in material constants, the velocity and pressure fields are found to be similar in both theories in the case of a concentrated body force acting in an unbounded medium. Moreover, this pressure is identical to that obtained in the classical Navier-Stokes theory; hence the presence of couple stresses does not affect the pressure field.

References

1. T. ARIMAN, M. A. TURK, N. D. SYLVESTER, *Applications of microcontinuum fluid mechanics*, Int. J. Engng. Sci., **12**, 273, 1974.
2. K. A. KLINE, S. J. ALLEN, *Concentration effects in oscillatory blood flow*, Biorheology, **6**, 99, 1969
3. T. ARIMAN, M. A. TURK, N. D. SYLVESTER, *On steady and pulsatile flow of blood*, J. Appl. Mech., **41**, 1, 1974.
4. G. BUGALIARELLO, J. SEVILLA, *Velocity distribution and other characteristics of steady and pulsatile blood flow in fine glass tubes*, Biorheology, **7**, 85, 1970.
5. V. K. STOKES, *Couple stresses in fluids*, Phys. Fluids, **9**, 1709, 1966.
6. E. L. AERO, A. N. BULYGIN, E. V. KUVSHINSKII, *Asymmetricheskaia gidromekhanika*, Prikl. Mat. Mekh., **29**, 297, 1965.

7. N. SANDRU, *On some problems of the linear theory of asymmetric elasticity*, Int. J. Engng. Sci., **4**, 81, 1966.
8. K. L. CHOWDHURY, P. G. GLOCKNER, *Representations in elastic dielectrics*, Int. J. Engng. Sci., **12**, 597, 1974.
9. H. LAMB, *Hydrodynamics*, Cambridge Univ. Press, London 1974.
10. A. C. ERINGEN, *Theory of micropolar fluids*, J. Maths. Mech., **16**, 1, 1966.
11. J. L. BLEUSTEIN, A. E. GREEN, *Dipolar fluids*, Int. J. Engng. Sci., **5**, 323, 1967.
12. H. RAMKISSOON, S. R. MAJUMADAR, *Drag on an axially symmetric body in the Stokes flow of micropolar fluid*, Phys. Fluids, **19**, 16, 1976.

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