Arch. Mech., 38, 5–6, pp. 565–575, Warszawa 1986

Bubble flow considered as a continuous medium with a spherical microstructure(*)

J. S. DARROZES and G. MICHELET (PALAISEAU)

THE CLASSICAL micromorphic media theory is used to describe a bubbly flow without both phase change and slip velocity. In order to take into account the bubble growth only, a spherical microdeformation tensor is introduced. The virtual powers principle leads to the macro- and micro-momentum equations and the first principle is used to write down the energy equation when mean gas and liquid temperatures are assumed to be equal. The constitutive laws are derived from the second principle of thermodynamics and physical considerations give an equation of state which allows to close the full set of equations for such media.

Klasyczną teorię mikromorficzną zastosowano do opisu płynu z pęcherzykami przy założeniu braku przemian fazowych. Dla uwzględnienia wzrostu pęcherzyków wprowadzono tensor sferycznej mikrodeformacji. Zasada mocy wirtualnych prowadzi do równań makro- i mikropędów, a pierwsza zasada pozwala wyprowadzić równanie energii przy założeniu równości średnich temperatur gazu i cieczy. Prawa konstytutywne otrzymuje się z drugiej zasady termodynamiki, a rozważania fizyczne prowadzą do równania stanu, które pozwala zamknąć pełny układ równań rozpatrywanych ośrodków.

Классическая микроморфическая теория применена к описанию жидкости с пузырьками, при предположении отсутствия фазовых превращений. Для учета роста пузырьков введен тензор сферической микродеформации. Принцип виртуальных мощностей приводит к уравнениям макро- и микроимпульсов, а первое начало позволяет вывести уравнение энергии, при предположении равенства средних температур газа и жидкости. Определяющие законы получаются из второго начала термодинамики, а физические рассуждения приводят к уравнению состояния, которое позволяет замкнуть полную систему уравнений рассматриваемых сред.

1. Introduction

RESEARCH work on bubbly flows until 1972 has been reviewed in the Annual Review of fluid mechanics by VAN WIJNGAARDEN [1]. It appears that micromorphic media theory, first introduced by C. ERINGEN [2] and later formalized using the virtual powers principle by P. GERMAIN [3], is a powerful tool which has not been used as yet to achieve a description of these flows.

This paper is the first step of a tentative marriage of bubbly flows with micromorphic media. In this theory, the fluid containing the bubbles is replaced by a macroscopic continuous medium and the microstructures, taken into account through a refined kinematic description of the mean flow, lead to non-Newtonian properties of the continuum. It is assumed that there is no slip velocity and that no phase change occurs. The constitutive

(*) Paper given at XVII Symposium on Advanced Problems and Methods in Fluid Mechanics, Sobieszewo, 2–6 September 1985.

laws are obtained, as usual, from the Clausius–Duhem inequality, without assuming that the entropy flux is equal to the heat flux divided by the temperature. Physical considerations on the microscopic scale of fluid properties lead to the expression of an equation of state when the liquid compressibility is neglected and when gas and liquid are supposed to have the same mean temperature θ.

2. Interpretation of a bubble flow as a micromorphic medium

In order to define mean quantities for the gas-liquid mixtures considered in this paper, it is required that the microscopic scale, denoted by εL, is much smaller than a characteristic length L of the macroscopic evolution (for instance a throat diameter or a profile chord). The mean quantities are then defined on an intermediate scale $\mu(\varepsilon)L$ with:

$$\varepsilon = \mu(\varepsilon) \leqslant 1,$$

when ε goes to zero. So, a volume D^* of the liquid-gas mixture with a finite size when measured with the scale $\mu(\varepsilon)L$ is a vanishing volume element dv when measured with the macroscopic scale L, but contains an infinitely large number of bubbles of order $(\mu/\varepsilon)^3$. For instance, the following mean quantities are involved in this paper:

a) void fraction: α

$$\alpha\, dv = \int_{D^*} 1_g^* dv^*$$

with $1_g^* = 1$ in the gas and 0 in the liquid;

b) density: ϱ

$$\varrho\, dv = \int_{D^*} \varrho^* dv^* = [\alpha\varrho_g + (1-\alpha)\varrho_L];$$

where indice g and L apply to gas and liquid, respectively;

c) microinertia tensor: $\mathbf{I}$

$$\varrho \mathbf{I}\, dv = \int_{D^*} \varrho^* \mathbf{x}^* \mathbf{x}^* dv^*,$$

where $\mathbf{x}^*$ denotes the local position of a point $\mathbf{M}^*$ with respect to the center $\mathbf{M}$ of the domain D^*.

The kinematic description of the medium is given by: i) the mean macroscopic velocity $\mathbf{U}(\mathbf{M}, t)$ ii) a local deformation tensor $\boldsymbol{\chi}(\mathbf{M}, t)$ defined by $\mathbf{U}(\mathbf{M}^*) = \mathbf{U}(\mathbf{M}) + \boldsymbol{\chi}(\mathbf{M})\mathbf{x}^*$.

In such a description, the exact microscopic motion does not appear. It is replaced by a mean local motion which is convected with the mean global velocity of D^*.

We restrict our attention to spherical micromorphic media for which, by definition, the microdeformation tensor is spherical, and involves only one scalar χ: $\boldsymbol{\chi} = \chi\mathbf{1}$. In that case, only the voluminal evolution due to bubble growth can be taken into account, and it is easily shown that the non-compressibility of the liquid phase leads to the following result:

$$[d(dv_g^*)/dt]/dv^* = [d(dv^*)/dt]/dv^* \quad \text{and} \quad 3\chi = (d\alpha/dt)/(1-\alpha) \text{ with } dv_g^* = \alpha dv.$$

Furthermore, assuming that the macroscopic compressibility is due only to the microscopic one, then:

$$\mathrm{Div}\,\mathbf{U} = 3\chi = (d\alpha/dt)/(1-\alpha).$$

3. Momentum equations

They are derived from the virtual power principle. Denoting by $\hat{\mathbf{U}}$ and $\hat{\chi}$ the virtual velocity and the microdeformation scalar, the virtual power of internal forces inside a domain D is written as a linear form of the virtual elements of the kinematic description. The simplest frame-indifferent form is:

$$\hat{\mathbf{P}}_i = -\int_D (\mathbf{\Sigma}:\hat{\mathbf{D}}+s\hat{\chi})\,dv, \tag{3.1}$$

where $\mathbf{D}$ is the symmetric part of $\nabla\mathbf{U}$, $\mathbf{\Sigma}$ is the global stress tensor, and s is the microscopic stress scalar. Integrating (3.1) by parts, one can write:

$$\hat{\mathbf{P}}_i = \int_D (\hat{\mathbf{U}}\,\mathrm{Div}\,\mathbf{\Sigma}-s\hat{\chi})\,dv - \int_{\partial D} (\mathbf{\Sigma}\mathbf{n})\hat{\mathbf{U}}\,da, \tag{3.2}$$

where $\mathbf{n}$ is the unit vector normal to ∂D.

The virtual power of external forces is written in a similar form:

$$\hat{\mathbf{P}}_e = \int_D (\mathbf{f}\cdot\hat{\mathbf{U}})\,dv + \int_{\partial D} (\mathbf{T}\cdot\hat{\mathbf{U}})\,da, \tag{3.3}$$

where $\mathbf{f}$ is the force per unit volume acting on D and $\mathbf{T}$ is the force per unit area on the domain boundary ∂D, due to the complementary part of the medium located outside D. The virtual power of inertia forces requires the definition of a local specific density p_a:

$$\mathbf{P}_a = \int_D \varrho p_a\,dv \tag{3.4}$$

with

$$\varrho p_a\,dv = \int_{D^*} \varrho^*(d\mathbf{v}^*/dt)\hat{\mathbf{v}}^*\,dv.$$

In this expression, the local velocities $\mathbf{v}^*$ and $\hat{\mathbf{v}}^*$ on the intermediate scale $\mu(\varepsilon)L$ are related to the kinematic description through the relations:

$$\mathbf{v}^* = \mathbf{U}+\chi\mathbf{x}^* \quad \text{and} \quad \hat{\mathbf{v}}^* = \hat{\mathbf{U}}+\hat{\chi}\mathbf{x}^*. \tag{3.5}$$

Inserting (3.5) in (3.4), and because of the definition of the volume center $\mathbf{M}$:

$$\int_{D^*} \mathbf{x}^*\,dv^* = 0,$$

one obtains the following expression:

$$\hat{\mathbf{P}}_a = \int_D [\varrho\boldsymbol{\gamma}\hat{\mathbf{U}}+\varrho I(\chi^2+\chi)\hat{\chi}]\,dv, \tag{3.6}$$

where I is the trace of the microinertia tensor $\mathbf{I}$, and $\boldsymbol{\gamma} = d\mathbf{U}/dt$ is the mean acceleration of the liquid-gas mixture.

Formulae (3.2), (3.3) and (3.6) are used to give an explicit form of the virtual powers principle:

$$\hat{\mathbf{P}}_a = \hat{\mathbf{P}}_e + \hat{\mathbf{P}}_i.$$

The volume integrals lead to:

a macromomentum equation

$$\varrho\boldsymbol{\gamma} = \mathbf{f} + \mathrm{Div}\,\boldsymbol{\Sigma}, \tag{3.7}$$

a micromomentum equation

$$\varrho I(\chi + \chi^2) = -s \tag{3.8}$$

and the surface terms give the usual natural boundary condition $\mathbf{T} = \boldsymbol{\Sigma}\cdot\mathbf{n}$.

In order to complete the set of equations it is necessary to add:

a macroscopic continuity equation

$$d\varrho/dt + \mathrm{Div}\,\mathbf{U} = 0, \tag{3.9}$$

a microinertia balance equation

$$dI/dt = 2\chi I, \tag{3.10}$$

an equation for the absence of phase change, which means that the quality x is a constant

$$dx/dt = 0, \tag{3.11}$$

and an energy equation derived, as usual, from the expression of the first principle of thermodynamics

$$d(E+K)/dt = \mathbf{P}_e + Q, \tag{3.12}$$

where K is the kinetic energy, $\mathbf{P}_e$ the power of external forces, and Q is the rate of heat supply brought to the medium per unit of time. Combining (3.12) with the kinetic energy theorem,

$$dK/dt = \mathbf{P}_e + \mathbf{P}_i \tag{3.13}$$

it is easily found that $dE/dt = Q - \mathbf{P}_i$. Using the expression of the internal energy E for a domain D of the mixture,

$$E = \int_D \varrho e\, dv$$

and the expression relating Q to the heat supply r within the volume D and the heat flux $\mathbf{q}$ through the boundary ∂D,

$$Q = \int_D r\, dv - \int_{\partial D} \mathbf{q}\cdot\mathbf{n}\, da$$

one gets, without difficulties, the energy equation:

$$\varrho\, de/dt = -\mathrm{Div}\,\mathbf{q} + \boldsymbol{\Sigma}:\mathbf{D} + s\chi + r. \tag{3.14}$$

The proof of (3.9) and (3.10) is not given here, but these well-known results can be found in P. GERMAIN [2]. The expressions of $\mathbf{\Sigma}$, s, $\mathbf{q}$ in (3.7), (3.8), and (3.14) are given in the following paragraph devoted to the medium rheology. The expression of the specific internal energy e (Eq. (3.14)) is obtained from the equation of state introduced in Sect. 5.

4. Constitutive laws

The rheological properties of the medium are deduced from the second principle of thermodynamics written as a balance equation for the entropy η:

$$\varrho d\eta/dt = -\mathrm{Div}(\mathbf{h}/\theta)+(r/\theta)+\varrho\sigma, \tag{4.1}$$

where the entropy production σ is a non-negative quantity. The entropy flux $\mathbf{h}/\theta$ is not supposed to be the heat flux $\mathbf{q}$ divided by the mean temperature θ (which is assumed to be the same for both the liquid and gas). In the following, the Coleman and Noll procedure is used as it has been done by I. MÜLLER [3] for gas mixtures.

First, the voluminal heat supply r is eliminated in a combination of equation (3.14) and (4.1). Letting $\psi = e-\theta\eta$ the free energy, and $\mathbf{k}$, the vector $\mathbf{h}-\mathbf{q}$, one gets the so-called "full Clausius-Duhem inequality":

$$-\varrho(d\psi/dt+\eta d\theta/dt)+\mathbf{\Sigma}:\mathbf{D}+s\chi+\mathrm{Div}\,\mathbf{k}-\mathbf{h}\nabla\theta/\theta \le 0. \tag{4.2}$$

Then, we define a "first order Stokesian spherically micromorphic fluid" as a fluid for which all rheological and thermodynamical properties depend on all the objective unknowns found in equations (3.9), (3.10), (3.11), (3.7), (3.8), (3.14), and on their first gradients (first order fluid).

Next, the free energy ψ is written in a "*a-priori*" form of an equation of state which involves 23 scalar variables X_l, $\mathbf{l} = 1, \ldots, 23$:

$$\psi(\varrho, \theta, I, x, \nabla\varrho, \nabla\theta, \nabla I, \nabla x, \mathbf{D}, \chi),$$

and all other unknown functions, related to the

dynamic behaviour: $\mathbf{\Sigma}(X_\mathrm{l})$, $s(X_\mathrm{l})$

thermal properties: $\mathbf{q}(X_\mathrm{l})$, $\mathbf{k}(X_\mathrm{l})$

are supposed to depend on the same 23 arguments X_l.

In order to be consistent with the level of this simplest kinematic description, $\nabla\chi$ is not listed in the (X_l) because it is the gradient of a rate deformation tensor $\chi\mathbf{1}$, and it should be on the same level as $\nabla\mathbf{D}$ which is not present in this first gradient theory.

The technique used can be summarized as follows

a) In (4.2), $d\psi/dt$ is expressed in the form

$$d\psi/dt = (\partial\psi/\partial X_\mathrm{l})(dX_\mathrm{l}/dt). \tag{4.3}$$

b) Among all the dX_l/dt, $d\varrho/dt$; $d\nabla\varrho/dt$; dx/dt; $d\nabla x/dt$; dI/dt; $d\nabla I/dt$ can be expressed in terms of X_l, using equations (3.9), (3.11) and (3.10):

$$d\varrho/dt = -\varrho\,\mathrm{Div}\,U = -\varrho\delta_{ij}D_{ij} \tag{4.4}$$

from (4.9)

$$d\varrho_{,i}/dt = -\varrho_{,i}\delta_{ij}D_{ij}-\varrho U_{k,ki}-U_{j,i}\varrho_{,j};$$

$$dx/dt = 0 \tag{4.5}$$

from (4.11)

$$dx_{,J}/dt = -U_{J,i}x_{,J};$$

(4.6) $$dI/dt = 2\chi I$$

from (4.10)

$$dI_{,i}/dt = 2I\chi_{,i}+2\chi I_{,i}-I_{,k}D_{ki}-I_{,k}\Omega_{ki},$$

In these expressions, the Einstein summation convention is used, and any symbol $A_{,i}$ denotes the x_i derivative of A with respect to Cartesian coordinates x_1, x_2, x_3. For instance:

$$\Omega_{ki} = (U_{i,k}-U_{k,i})/2,$$
$$D_{ki} = (U_{i,k}+U_{k,i})/2.$$

c) The term $\mathrm{Div}\,\mathbf{k} = k_{i,i}$ in (4.2) is written

(4.7) $$\mathrm{Div}\,\mathbf{k} = (\partial k_i/\partial X_I)X_{I,i}$$

and contains 69 scalar terms.

d) By inserting (4.4), (4.5), (4.6) into (4.3), a new expression is obtained, which is next inserted in (4.2) together with (4.7). One thus obtains the full Clausius-Duhem inequality which contains 113 scalar terms to be found in the Annex, and written shortly here in the form

(4.8) $$CD \geq 0.$$

Now, the CD expression is reduced as follows:

i) CD involves dX_I/dt except $d\varrho/dt$; $d\varrho_{,i}/dt$; dx/dt; $dx_{,i}/dt$; dI/dt; $dI_{,i}/dt$ given by (4.4), (4.5) and (4.6). Inequality (4.8) must be valid whatever the medium history is, so, a term as (see the Annex)

$$-\varrho(\partial\psi/\partial\theta+\eta)d\theta/dt$$

must vanish for any $d\theta/dt$ and leads to the result:

$$\eta = -\partial\psi/\partial\theta.$$

It is easily shown, using the same argument, that

$$\partial\psi/\partial\theta_{,i} = \partial\psi/\partial I_{,i} = \partial\psi/\partial D_{ij} = \partial\psi/\partial\chi = 0$$

so, the free energy ψ is dependent on the following variables only:

$$\psi(\varrho, \theta, I, x, \nabla\varrho, \nabla I, \nabla x).$$

ii) The expression of CD (see the Annex) is linear in the higher order derivatives $\varrho_{,ij}$; $\theta_{,ij}$; $I_{,ij}$; $D_{jk,i}$; $\chi_{,i}$. The inequality must be valid for any spatial evolution, so one deduces that their coefficient must vanish:

(4.9) $$\partial k_i/\partial\varrho_{,j}+\partial k_j/\partial\varrho_{,i} = 0$$

as well as many others given in the Appendix. From the dependence of CD. upon $D_{jk,i}$ and $\chi_{,i}$:

(4.10) $$\partial k_i/\partial D_{jk}+\varrho^2\partial\psi/\partial\varrho_{,i}\delta_{jk} = 0,$$

(4.11) $$\partial k_i/\partial\chi-2\varrho I\partial\psi/\partial I_{,i} = 0.$$

iii) The non-objective term Ω_{ij} in CD must vanish, and this gives the following relations:

$$(\partial\psi/\partial\varrho_{,i})\varrho_{,k}-(\partial\psi/\partial\varrho_{,k})\varrho_{,i}=0,$$
$$(\partial\psi/\partial I_{,i})I_{,k}-(\partial\psi/\partial I_{,k})I_{,i}=0,$$
$$(\partial\psi/\partial x_{,i})x_{,k}-(\partial\psi/\partial x_{,k})_{,i}=0.$$

iv) A tedious integration of all these relations leads to the following result:

- $$\mathbf{k}=\mathbf{h}-\mathbf{q}=\nabla\varrho\Lambda(E_1\nabla\theta+E_2\nabla I+E_3\nabla x)+\nabla\theta\Lambda(E_4\nabla I+E_5\nabla x)+E_6(\nabla I\Lambda\nabla x)+(\mathbf{F}_1\Lambda\nabla\varrho)+(\mathbf{F}_2\Lambda\nabla\theta)+(\mathbf{F}_3\Lambda\nabla I)+\mathbf{F}_4\Lambda\nabla x)+\mathbf{C},$$

where the E_i, $\mathbf{F}_j$ and $\mathbf{C}$ are functions of ϱ, θ, I, and x only.

- $$\psi=|\nabla x|^2\phi(\varrho,\theta,I,x)+\Psi(\varrho,\theta,I,x).$$

v) Using the principle of frame indifference, the vector $\mathbf{k}$ must be equal to zero and $\mathbf{h}=\mathbf{q}$.

The existence of $|\nabla x|^2$ in the free energy leads to the great difficulties in the research of the equation of state. The mixture is supposed to be a "simple macro-fluid" at the end of this paper. This means, with the I. Muller's definition, that ψ does not depend on $|\nabla x|$ ([4)], and the reduced Clausius-Duhem inequality is thus obtained:

$$D_{ij}(\varrho^2\partial\psi/\partial\varrho\delta_{ij}+\Sigma_{ij})+\chi(s-2\varrho I\partial\psi/\partial I)-q_i\theta_i/\theta\geq 0. \tag{4.12}$$

At equilibrium, $D_{ij}=0$, $\chi=0$, and $\theta_{,i}=0$ must have double roots, so: $\mathbf{\Sigma}^{(e)}=-p\mathbf{1}$ where $p=-\varrho^2\partial\psi/\partial\varrho$ is the macroscopic pressure, $s^{(e)}=-\pi$ where $\pi=2\varrho l\partial\psi/\partial l$ is called "extra pressure", and $\mathbf{q}^{(e)}=0$. $\mathbf{\Sigma}$ and s are split into two parts: the value at equilibrium and a dissipative term.

$$\mathbf{\Sigma}=-p\mathbf{1}+\mathbf{T}\quad\text{and}\quad s=-\pi+\tau \tag{4.13}$$

and the Clausius-Duhem inequality involves only dissipations:

$$\mathbf{T}:\mathbf{D}+\tau\chi-\nabla\theta/\theta\geq 0. \tag{4.14}$$

Using the Onsager principle, and assuming the isotropy of the medium, the constitutive laws are obtained through a very classical procedure, shortly reminded below for the viscous stress tensor $\mathbf{T}$.

First, $\mathbf{T}$ is written as a linear form of $\mathbf{D}$, χ, and $\nabla\theta$:

$$\mathbf{T}=\mathbf{L}^{(1)}(\mathbf{D})+\mathbf{L}^{(2)}(\chi)+\mathbf{L}^{(3)}(\nabla\theta). \tag{4.15}$$

Then, the principle of frame indifference is used. Changing the space orientation by changing the orientation of each unit vector of a reference frame, it is easily seen that $\mathbf{D}$ and χ are invariant, and that only $\nabla\theta$ has the opposite sign. This proves that $\mathbf{L}^{(3)}(\nabla\theta)=0$. Then, using the assumption of isotropy, $\mathbf{L}^{(1)}$ must involve only a multiplicative constant or a linear invariant of $\mathbf{D}$, which is Div $\mathbf{U}$, so:

$$\mathbf{T}=\lambda(\mathrm{Div}\,\mathbf{U})\mathbf{1}+2\mu\mathbf{D}+\gamma\chi. \tag{4.16}$$

In a similar way, the following laws are obtained:

$$\tau=\alpha\,\mathrm{Div}\,\mathbf{U}+\beta\chi, \tag{4.17}$$
$$\mathbf{q}=-k\nabla\theta. \tag{4.18}$$

5. The equation of state

It is not possible to know the function $\psi(\varrho, \theta, I, x)$ or $e(\varrho, \eta, I, x) = \psi - \theta\eta$ without a physical analysis of the bubbly flow. It is known that, on the microscopic scale εL, only the gas pressure p_g inside the bubbles has a thermodynamic meaning: $p_g = \varrho_g r\theta$, if the gas behaves as an ideal gas. The pressure p_L inside the liquid phase has no thermodynamical meaning because of the non-compressibility of the liquid. On the macroscopic scale L, two kinds of pressures have been pointed out:

the mean total pressure $p = \varrho^2 \partial\psi/\partial\varrho$,

the extra-internal pressure $\pi = -2\varrho I \partial\psi/\partial I$.

One of these two pressures has no thermodynamical meaning, so, writing again the expression of the virtual power of internal forces in a slightly different form than (3.1), and taking into account (4.13) one gets:

$$\mathbf{P}_i = \int [(p+\pi/3)\operatorname{Div}\mathbf{U} + \pi(3\chi - \operatorname{Div}\mathbf{U})]\,dv + \text{dissipative terms}. \tag{5.1}$$

From the non-compressibility of the liquid, one deduces, as it has been said in Sect. 1 the internal liaison:

$$\operatorname{Div}\mathbf{U} = 3\chi. \tag{5.2}$$

Expression (5.1) shows that π is a Lagrangian multiplier associated with the internal liaison (5.2) and its value can be determined only after solving the full set of equations governing the flow motion.

From: $d\psi = (\partial\psi/\partial\varrho)d\varrho + (\partial\psi/\partial\theta)d\theta + (\partial\psi/\partial I)dI + (\partial\psi/\partial x)dx$, and taking into account that no phase change occurs ($dx = 0$), the total differential of the specific internal energy is written:

$$de = \theta\, d\eta + (p/\varrho^2)d\varrho - (\pi/2\varrho I)dI. \tag{5.3}$$

This expression can be considered as the addition to:

(5.4)
a specific reversible heat supply: $d'Q = \theta d\eta$;
a specific reversible work received: $d'W = (p/\varrho^2)d\varrho - (\pi/2\varrho I)dI.$

Using relations (3.2), (3.9), (3.10) and $3\chi = (d\alpha/dt)/(1-\alpha)$, it is easily shown that the density and the microinertia depend only on the void fraction:

$$d\varrho/\varrho = -d\alpha/(1-\alpha), \tag{5.5}$$

$$dI/I = 2d\alpha/3(1-\alpha). \tag{5.6}$$

Inserting these expressions into (5.4), the specific reversible work is written:

$$d'W = -(p+\pi/3)d\alpha/\varrho(1-\alpha). \tag{5.7}$$

A direct evaluation of $d'W$ can be obtained with a rustic reasoning which is classical in thermostatics. Let us consider a tube closed at one end, with a moving piston at the other end, filled with a mass M_g of gas and a mass M_L of liquid occupying the volumes V_g and V_L, respectively. For an infinitesimal piston displacement, only the gas phase is working and one can write

$$d'W = -p_g d[V_g/(M_L+M_g)] = -p_g d[V_g/\varrho(V_L+V_g)] = -p_g d(\alpha/\varrho),$$

and $d(\alpha/\varrho) = d\alpha/\varrho - (\alpha/\varrho^2)(d\varrho/d\alpha)d\alpha.$

Taking into account Eq. (5.5), per unit of the total mass one obtains a new expression of the specific work:

$$d'W = -p_g d\alpha/\varrho(1-\alpha). \tag{5.8}$$

By comparison with Eq. (5.7), the gas pressure p_g is related to the total mean pressure p and the extra-pressure π:

$$p_g = p+\pi/3. \tag{5.9}$$

Assuming the ideal gas behaviour for bubbles, $p_g = \varrho_g r\theta$, and taking into account the relation: $\varrho = \alpha\varrho_g+(1-\alpha)\varrho_L$ where ϱ_L is a known constant quantity for the incompressible liquid, one obtains from Eq. (5.8) the definite expression of the equation of state:

$$p+\pi/3 = r\theta[\varrho-(1-\alpha)\varrho_L]/\alpha. \tag{5.10}$$

A relationship involving the extra-pressure π, the local gas pressure p_g and the liquid pressure p_L is deduced from Eq. (5.9), and from the relation: $p = \alpha p_g+(1-\alpha)p_L$. It is found that:

$$\pi = 3(1-\alpha)(p_g-p_L).$$

It is interesting to notice that π which has no thermodynamical meaning, as it has been seen previously, represents the dynamical effect of bubble growth. In a quasi-static evolution, the interface equilibrium requires (when surface tension is neglected) $p_g = p_L$, and $\pi = 0$.

The energy equation (3.14) has a new expression, taking into account the internal liaison and the constitutive laws:

$$\varrho\, de/dt = \mathrm{Div}(k\nabla\theta)-(p+\pi/3)\mathrm{Div}\,\mathbf{U}+\mathbf{T}:\mathbf{D}+\tau\chi+r. \tag{5.11}$$

From

$$de = d'Q+d'W = \theta\, d\eta-(p+\pi/3)/d\alpha/\varrho(1-\alpha),$$

one can write: $de = c\,d\theta$, where c is the specific heat at constant void fraction α and quality x.

6. Concluding remarks

The micromorphic media theory has been applied to describe the bubbly flow motions without phase change. The constitutive laws given in Sect. 4 and the equation of state obtained in Sect. 5 lead to the full set of equations governing the liquid-gas mixture, considered as a continuous medium with a spherical microstructure. The entropy production can be expressed, by taking into account equation (5.4), (5.7), and (5.10):

$$\varrho\theta\sigma = \mathbf{T}:\mathbf{D}+\tau\chi-\mathbf{q}\nabla\theta/\theta. \tag{6.1}$$

For instance, a non-dissipative spherically micromorphic fluid, is governed by the following equations:

mass: $d\varrho/dt+\varrho\,\mathrm{Div}\,\mathbf{U} = 0,$

phase change: $dx/dt = 0,$

$$
\begin{aligned}
&\text{microinertia:} && dI/dt-2\chi I = 0,\\
&\text{macromomentum:} && \varrho d\mathbf{U}/dt+\nabla p = 0,\\
&\text{micromomentum:} && \varrho I(d\chi/dt+\chi^2)-\pi = 0,\\
&\text{energy:} && \varrho c(d\theta/dt)+\operatorname{Div}\mathbf{U}(p+\pi/3) = 0,\\
&\text{equation of state:} && p+\pi/3-r\theta[\varrho-(1-\alpha)\varrho_L]/\alpha = 0.
\end{aligned}
$$

One must add the internal liaison:

$$\operatorname{Div}\mathbf{U} = 3\chi = (d\alpha/dt)/(1-\alpha)$$

and an obvious expression of the specific heat c:

$$c = xc_g+(1-x)c_L.$$

Annex

The full Clausius-Duhem inequality has the following expression:

$$
\begin{aligned}
&-\varrho(d\theta/dt)(\eta+\partial\psi/\partial\theta)-\varrho(d\theta_{,i}/dt)(\partial\psi/\partial\theta_{,i})-\varrho(dD_{ij}/dt)(\partial\psi/\partial D_{ij})-\varrho(d\chi/dt)(\partial\psi/\partial\chi)\\
&\quad+(\partial k_i/\partial\varrho_{,j})\varrho_{,ij}+(\partial k_i/\partial\theta_{,j})\theta_{,ij}+(\partial k_i/\partial I_{,j})I_{,ij}+(\partial k_i/\partial x_{,j})x_{,ij}+[\partial k_i/\partial D_{jk})\\
&\quad+\varrho^2\partial\psi/\partial\varrho_{,i}]D_{jk,i}+(\partial k_i/\partial\chi)\chi_{,i}+\varrho\Omega_{ki}[(\partial\psi/\partial\varrho_{,i})\varrho_{,k}+(\partial\psi/\partial I_{,i})I_{,k}+(\partial\psi/\partial x_{,i})x_{,k}]\\
&\quad+(\partial k_i/\partial\varrho)\varrho_{,i}+(\partial k_i/\partial\theta)\theta_{,i}+(\partial k_i/\partial I)I_{,i}+(\partial k_i/\partial x)x_{,i}-2\varrho(\partial\psi/\partial I_{,i})(I\chi_{,i}+\chi I_{,i})\\
&\quad+\varrho D_{ij}[\varrho(\partial\psi/\partial\varrho)\delta_{ij}+\varrho_{,k}(\partial\psi/\partial\varrho_{,k})\delta_{ij}+\varrho_{,j}(\partial\psi/\partial\varrho_{,i})+I_{,j}(\partial\psi/\partial I_{,i})+x_{,j}(\partial\psi/\partial x_{,i})\\
&\quad+\Sigma_{ij}]-2\varrho I\chi\partial\psi/\partial I+s\chi-k_i\theta_{,i}/\theta \geqq 0.
\end{aligned}
$$

The linear term $d\theta/dt$ leads to $\eta = -\partial\psi/\partial\theta$.

The linear terms $d\theta_{,j}/dt$; dD_{ij}/dt; $d\chi/dt$ lead to

$$\psi = \psi(\varrho, \theta, I, x, \nabla\varrho, \nabla I, \nabla x).$$

The linear term $\varrho_{,ij}$ leads to the relations $\partial k_i/\partial\varrho_{,j}+\partial k_j/\partial\varrho_{,i} = 0$,
The linear term $\theta_{,ij}$ leads to the relations $\partial k_i(\partial\theta_{,j}+\partial k_j/\partial\theta_{,i} = 0$.
The linear term $I_{,ij}$ leads to the relations $\partial k_i/\partial I_{,j}+\partial k_j/\partial I_{,i} = 0$.
The linear term $x_{,ij}$ leads to the relations $\partial k_i/\partial x_{,j}+\partial k_j/\partial x_{,i} = 0$.
The linear term $D_{jk,i}$ leads to the relations

$$\partial k_i/\partial D_{jk}+\varrho^2(\partial\psi/\partial\varrho_{,i}) = 0.$$

The linear term $\chi_{,i}$ leads to the relations

$$\partial k_i/\partial\chi-2\varrho I\partial\psi/\partial I_{,i} = 0.$$

The non-objective term Ω_{ki} requires the relations:

$$
\begin{aligned}
(\partial\psi/\partial\varrho_{,i})\varrho_{,k}-(\partial\psi/\partial\varrho_{,k})\varrho_{,i} &= 0,\\
(\partial\psi/\partial I_{,i})I_{,k}-(\partial\psi/\partial I_{,k})I_{,i} &= 0,\\
(\partial\psi/\partial x_{,i})x_{,k}-(\partial\psi/\partial x_{,k})x_{,i} &= 0.
\end{aligned}
$$

References

1. L. VAN WIJNGAARDEN, *One-dimensional flow of liquids containing small gas bubbles*, Annual Review of Fluid Mech., **4**, 369–396, 1972.
2. C. ERINGEN, *Simple microfluids*, International J. Enging. Sci., **2**, 205–217, 1964.
3. P. GERMAIN, *Methodes des puissances virtuelles en mécanique des milieux continus, Part II. Microstructures*, SIAM J. Appl. Mathem., **25**, 3, 1973.
4. I. MÜLLER, *A thermodynamic theory of mixtures of fluids*, Arch. Rational Mech. and Analysis, **28**, 1–139, 1968.
5. J. S. DARROZES et G. MICHELET, *Flux d'entropie des fluids micromorphiques*, C.R.A.S. Paris, t. 262, serie II, pp 1093–1096, Mai 1981.

UNIVERSITÉ PIERRE ET MARIE CURIE
UER DE MÉCANIQUE, PARIS
et
ECOLE NATIONALE SUPÉRIEURE DE TECHNIQUES AVANCÉES
CENTRE DE L'YVETTE, CHEMIN DE LA HUNIÈRE, PALAISEAU, FRANCE.

Received November 22, 1985.