22.

ON THE INTERSECTIONS, CONTACTS, AND OTHER CORRELATIONS OF TWO CONICS EXPRESSED BY INDETERMINATE COORDINATES.

[*Cambridge and Dublin Mathematical Journal*, v. (1850), pp. 262—282.]

LET $U=0$, $V=0$ be two homogeneous equations of the second degree with real coefficients, between the same three variables ξ, η, ζ.

The direct and most general mode of determining the intersections of the conics expressed by these equations would be to make

$$a\xi + b\eta + c\zeta = t,$$

$$a'\xi + b'\eta + c'\zeta = u:$$

eliminating ξ, η, ζ between the four equations in which they appear, there results a biquadratic equation between t and u. The nature of the intersections will depend upon the nature of the roots of this biquadratic; and thus the conditions may be expressed analytically, which will represent the several cases of all the intersections being real or all imaginary, or one pair real and the other imaginary. These analytical conditions will depend upon the signs of certain functions of the coefficients of the given and the *assumed* equations being of an assigned character; my endeavour has been to obtain conditions of a character perfectly symmetrical and free from the coefficients arbitrarily introduced.

In this research I have only partially succeeded, but the method employed, and some of the collateral results, will, I think, be found of sufficient interest to justify their appearance in the pages of this *Journal*.

Adopting Mr Cayley's excellent designation, let the four points of intersection of the two conics be called a quadrangle. This quadrangle will have three pairs of sides; the intersections of each pair, from principles of analogy, I call the vertices of the quadrangle. Then, inasmuch as the four sets of ratios $\xi : \eta : \zeta$, corresponding with the four sets of the ratio $t : u$, must be so related that we may always make

$$\frac{\xi_1}{\zeta_1} = a + b\sqrt{(-1)}, \qquad \frac{\eta_1}{\zeta_1} = c + d\sqrt{(-1)},$$

$$\frac{\xi_2}{\zeta_2} = a - b\sqrt{(-1)}, \qquad \frac{\eta_2}{\zeta_2} = c - d\sqrt{(-1)},$$

$$\frac{\xi_3}{\zeta_3} = \alpha + \beta\sqrt{(-1)}, \qquad \frac{\eta_3}{\zeta_3} = \gamma + \delta\sqrt{(-1)},$$

$$\frac{\xi_4}{\zeta_4} = \alpha - \beta\sqrt{(-1)}, \qquad \frac{\eta_4}{\zeta_4} = \gamma - \delta\sqrt{(-1)},$$

we may easily draw the following conclusions.

If all the four points of the quadrangle of intersection are real, the three vertices and the three pairs of sides are all real. If only two points of the quadrangle are real, one vertex and one of the three pairs of sides will be real; the other two vertices and two pairs of sides being imaginary. If all four points of the quadrangle are unreal, one pair of sides will be real and the other two pairs imaginary, as in the last case; but all the three vertices will remain real, as in the first case. Hence we have a direct and simple criterion for distinguishing the case of *mixed* intersection from intersection wholly real or wholly imaginary; namely, that the cubic equation of the roots of which the coordinates of the vertices are real linear functions shall have *a pair of imaginary roots*. This is the sole and unequivocal condition required.

The equation in question is, or ought to be, well known to be the determinant in respect to ξ, η, ζ of $\lambda U + \mu V$. In fact, if we write

$$U = a\xi^2 + b\eta^2 + c\zeta^2 + 2a'\eta\zeta + 2b'\zeta\xi + 2c'\xi\eta,$$

$$V = \alpha\xi^2 + \beta\eta^2 + \gamma\zeta^2 + 2\alpha'\eta\zeta + 2\beta'\zeta\xi + 2\gamma'\xi\eta,$$

$$\lambda U + \mu V = (a\lambda + \alpha\mu)\,\xi^2 + \&\text{c.} = A\xi^2 + B\eta^2 + C\zeta^2 + 2A'\eta\zeta + 2B'\zeta\xi + 2C'\xi\eta,$$

the ratios of the coordinates ξ, η, ζ of the vertex of $\lambda U + \mu V$ may easily be shown to be identical with

$$AB - C^2 : C'A' - B'B : B'C' - A'A,$$

and will be real or imaginary as $\lambda : \mu$ is one or the other.

If then the cubic equation in $\lambda : \mu$, namely, $\underset{\xi\eta\zeta}{\square}(\lambda U + \mu V) = 0$, has a pair of imaginary roots, that is, if $\underset{\lambda\mu}{\square}\,\underset{\xi\eta\zeta}{\square}(\lambda U + \mu V)$ is a positive quantity, the intersections of U and V are of a mixed kind, that is, the two conics have two real points in common.

I may remark here, *en passant*, that if we form the biquadratic equation in t and u, $\phi(t, u) = 0$ from the equations

$$U = 0,$$
$$V = 0,$$
$$a\xi + b\eta + c\zeta = t,$$
$$a'\xi + b'\eta + c'\zeta = u,$$

and if any reducing cubic of this equation be $P(\theta, \omega) = 0$, the determinant of $P(\theta, \omega)$ must, from what has been shown above, be identical with $\underset{\lambda\mu}{\square}\,\underset{\xi\eta\zeta}{\square}(\lambda U + \mu V)$ multiplied by some *squared function* of the extraneous coefficients

$$a, b, c; \; a', b', c'.$$

If $\square\square(\lambda U + \mu V)$ is a negative quantity, it remains to distinguish between the cases of the conics intersecting really in four points or not at all.

The most obvious mode of proceeding to distinguish between purely real and purely imaginary intersections would be as follows. Let λ_1, μ_1; λ_2, μ_2; λ_3, μ_3, be the three sets of values of λ, μ which satisfy the equation

$$\square(\lambda U + \mu V) = 0$$

and make

$$A_1 = a\lambda_1 + \alpha\mu_1, \qquad A_2 = a\lambda_2 + \alpha\mu_2, \qquad A_3 = a\lambda_3 + \alpha\mu_3,$$
$$C_1 = c\lambda_1 + \gamma\mu_1, \qquad C_2 = c\lambda_2 + \gamma\mu_2, \qquad C_3 = c\lambda_3 + \gamma\mu_3,$$
$$B_1' = b'\lambda_1 + \beta'\mu_1, \qquad B_2' = b'\lambda_2 + \beta'\mu_2, \qquad B_3' = b'\lambda_3 + \beta'\mu_3,$$
$$A_1C_1 - B_1'^2 = e_1, \qquad A_2C_2 - B_2'^2 = e_2, \qquad A_3C_3 - B_3'^2 = e_3.$$

Now if the equation

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2A'\eta\zeta + 2B'\zeta\xi + 2C'\xi\eta = 0$$

represent a pair of straight lines, it may be thrown into the form

$$Au^2 + \frac{AC - B'^2}{A}v^2 = 0,$$

where u and v are linear functions of ξ, η, ζ, and the straight lines will be real or imaginary, according as $B'^2 - AC$ is positive or negative; hence one or else all of the quantities e_1, e_2, e_3, will be necessarily negative, and the intersections will be all real or all imaginary, according as all three are negative or only one is so. A cubic equation in e may be formed containing e_1, e_2, e_3 as its roots by eliminating between the equations

$$e = AC - B'^2; \quad \square(\lambda U + \mu V) = 0,$$

and the conditions for the reality of the intersections will be that all four coefficients of this cubic shall be of the same sign, which in reality amount only to two, since the first and last must in all cases have the same sign.

The same objection however of want of symmetry and consequent irrelevancy and complexity attaches to this as much as to the method originally proposed. The following treatment of the question relieves the objection of want of symmetry as far as the coefficients of the same equation are concerned, but in its practical application necessitates an arbitrary and therefore unsymmetrical election to be made between the two sets of coefficients appertaining to the two equations. It is however, I think, too curious and suggestive to be suppressed.

I observe that if the four intersections are all real, an imaginary conic cannot be drawn through them; for the equation to an imaginary conic may always be reduced to the form $Ax^2+By^2+Cz^2=0$, where A, B, C are all positive and can therefore have at utmost one real point. Consequently the case of total non-intersection is distinguishable from that of complete intersection by the peculiarity that in the one case μ may be so taken that $U+\mu V=0$ shall represent an imaginary conic, that is, $U+\mu V$ will be a function whose sign never changes for real values of ξ, η, ζ, whereas in the latter case no value of μ will make $U+\mu V=0$ the equation to an imaginary conic, and therefore $U+\mu V$ will have values on both sides of zero. On the other hand, it is obvious that an infinite number of real as well as unreal conics may be drawn through four imaginary points of intersection. Consequently if we make $U+\mu V=0$ (supposing the intersections of U and V to be imaginary), there will be a range or ranges of values of μ consistent, and another range or ranges of values of μ inconsistent with real values of ξ, η, ζ; in other words, $U\pm\mu V=0$ treated as an equation between the four variables ξ, η, ζ, μ, will give one or more maxima or minima values of μ in the case supposed, but no such values when the intersections are two or all of them real.

To determine these values of μ, let $d\mu=0$; then we have

$$\frac{d}{d\xi}(U-\mu V)=0,$$

$$\frac{d}{d\eta}(U-\mu V)=0,$$

$$\frac{d}{d\zeta}(U-\mu V)=0,$$

that is $$\underset{\xi\eta\zeta}{\square}(U-\mu V)=0.$$

In order that any value of μ found from this equation may be a maximum or minimum, Lagrange's condition requires that

$$\left(h\frac{d}{d\xi}+k\frac{d}{d\eta}+l\frac{d}{d\zeta}\right)^2\mu$$

may be a function of unchangeable sign.

Now
$$\frac{dU}{d\xi} = \mu \frac{dV}{d\xi} + V \frac{d\mu}{d\xi},$$

therefore since $d\mu = 0$,
$$\frac{d^2U}{d\xi^2} = \mu \frac{d^2V}{d\xi^2} + V \frac{d^2\mu}{d\xi^2}.$$

Hence
$$\frac{d^2\mu}{d\xi^2} = \frac{1}{V}\left(\frac{d}{d\xi}\right)^2 \{U - \mu V\};$$

similarly
$$\frac{d}{d\xi} \cdot \frac{d}{d\eta} = \frac{1}{V} \frac{d}{d\xi} \cdot \frac{d}{d\eta} \{U - \mu V\},$$

&c. &c. &c.

Making now as before
$$U = a\xi^2 + b\eta^2 + \text{\&c.},$$
$$V = \alpha\xi^2 + \beta\eta^2 + \text{\&c.},$$
$$a - \mu\alpha = A, \quad b - \mu\beta = B, \text{ \&c.},$$

the condition for μ, a root of $\square \{U - \mu V\} = 0$, giving μ a maximum or minimum, may be expressed by saying that
$$Ah^2 + Bk^2 + Cl^2 + 2A'kl + 2B'hl + 2C'hk$$
shall be unchangeable in sign for all real values of h, k, l.

The above quantity, by virtue of the equation $\square = 0$, is always the product of two linear functions. Hence we see, as above indicated, that if all these pairs are real, that is, if all the points of intersection of U and V are real, there is no maximum or minimum value of μ; but if only one pair be real and the other two pairs be imaginary, that is, if all the four intersections are imaginary, then two of the values of μ, namely those corresponding to the imaginary pairs, are real maxima or minima values of μ, but the third is illusory.

Now I shall show that if $V = 0$ is a *real* conic, but the intersections of U and V are all unreal, the value of μ which makes $U + \mu V$ the product of real linear functions of ξ, η, ζ, is always one or the other *extreme* of the three values of μ which satisfy the equation
$$\square (U - \mu V) = 0.$$

Assume as the three axes of coordinates the three lines joining the vertices of the quadrangle each with each, the two non-intersecting conics may evidently be written under the form
$$U = c(x^2 + y^2) - e(y^2 + z^2) = 0,$$
$$V = -\gamma(x^2 + y^2) + \epsilon(y^2 + z^2) = 0;$$

these equations being only other modes of writing

$$U = Ax^2 + By^2 + Cz^2,$$

$$V = A'x^2 + B'y^2 + C'z^2,$$

in which A, B, C; A', B', C' will be real, because by hypothesis $\square(U+\mu V)=0$ has all its roots real.

Hence x, y, z are linear functions of ξ, η, ζ, and consequently, by a simple inference from a theorem of Prof. Boole*, the roots of $\underset{\xi\eta\zeta}{\square}\{U+\mu V\}$ are identical with those of

$$\underset{xyz}{\square}\{U+\mu V\} = 0.$$

These latter are evidently $\frac{c}{\gamma}$, $\frac{e}{\epsilon}$, $\frac{c-e}{\gamma-\epsilon}$; the third of which is the one which makes $U+\mu V$ the product of two *real* linears, for we have

$$\gamma U + cV = (c\epsilon - \gamma e)(y^2 + z^2),$$

$$\epsilon U + eV = (\epsilon c - e\gamma)(x^2 + y^2),$$

$$(\gamma - \epsilon)\, U + (c - e)\, V = (c\epsilon - e\gamma)(z^2 - x^2)\dagger.$$

Now
$$\frac{c}{\gamma} - \frac{c-e}{\gamma-\epsilon} = \frac{e\gamma - c\epsilon}{\gamma(\gamma-\epsilon)},$$

$$\frac{e}{\epsilon} - \frac{c-e}{\gamma-\epsilon} = \frac{e\gamma - c\epsilon}{\epsilon(\gamma-\epsilon)};$$

and ϵ, γ are supposed to have the same sign, as otherwise V would be an unreal conic; hence the ascending or descending order of magnitudes of the three values of λ follows the scale $\frac{c}{\gamma}$, $\frac{e}{\epsilon}$, $\frac{c-e}{\gamma-\epsilon}$, as was to be shown.

Imagine now lengths reckoned on a line corresponding to all values of μ from $-\infty$ to $+\infty$, and mark off upon this line by the letters A, B, C, the lengths corresponding with the three roots of $\square(U+\mu V)=0$. Then observing that when $\mu = \pm\infty$, $U+\mu V$ is of the same nature as V, and is therefore a possible conic by hypothesis, and agreeing to understand by a possible and impossible region of μ, a range of values for which $U+\mu V$ corresponds to a possible and impossible conic respectively, one or the other of the annexed schemes will represent the circumstances of the case supposed:

$-\infty$ ☜ Poss. Reg. A Imposs. Reg. B Poss. Reg. C Poss. Reg. ☞ $+\infty$

$-\infty$ ☜ Poss. Reg. A Poss. Reg. B Imposs. Reg. C Poss. Reg. ☞ $+\infty$

But in either scheme it is essential to observe that the *middle* root of $\square(U+\mu V)=0$ divides a possible from an impossible region; and therefore

* See Postscript.

† $z^2 - x^2 = 0$ of course represents a *real* pair of lines.

if we can find n, ν, any two values lying between the first and second and second and third roots of the above equation arranged in order of their magnitude, one of the two equations $U+\nu V=0$, $U+nV=0$, will represent a possible and the other an impossible conic: one such couple of values may always be found by taking the roots of the quadratic equation

$$\frac{d}{d\mu}\square\{U+\mu V\}=0.$$

Hence calling the two roots thereof m and M, we see (which is in itself a theorem) that one at least of the conics $U+mV=0$, $U+MV=0$, must be a possible conic, provided only that $V=0$ be a possible conic: if both $U+mV$ and $U+MV$ are possible conics, the intersections of U and V are all real, and if not, not*. The criteria for distinguishing possible from impossible conics being well known need not be stated in this place.

We may of course proceed analogously by forming the two conics $lU+V$, $LU+V$, where l and L are roots of $\frac{d}{d\lambda}\square\{\lambda U+V\}=0$ upon the supposition of $U=0$ being a possible conic.

If either of the two U and V be not possible, their intersections are of course impossible, and the question is already decided.

It will be seen as pre-indicated that this method only fails in symmetry because of the choice between the couples m, M, and l, L. But moreover a perfect method for the discrimination of the two cases of *unmixed* intersection one from the other should (perhaps?) require the application of only a single test (in lieu of the two conditions which the above method supposes), over and above the condition which expresses the fact of the intersections being so unmixed. Such more perfect method I have not yet been able to achieve.

Another interesting question of intersections remains to be discussed, namely, supposing the two conics are known to be non-intersecting, how are we to ascertain if they are external to one another, or if one contains the other? In order to settle this point we must first establish a criterion for determining whether a given *point* is internal or external to a given conic; the point being in general said to be external when two real tangents can be drawn from it to the curve, and internal when this cannot be done.

* It must be well observed however that the possibility of the conics $U+mV$ and of $U+MV$ does not imply the reality of the intersections unless the conic V is known to be possible.

For if V be impossible ϵ and γ have opposite signs, and therefore $\frac{c-e}{\gamma-\epsilon}$ is intermediate between $\frac{c}{e}$ and $\frac{\gamma}{\epsilon}$, and the scheme for μ will be as here annexed:

$-\infty$ Impossible. A Possible. B Possible. C Impossible. $+\infty$

so that $U+mV$ and $U+MV$ will both represent possible conics.

Let now

$$\phi(x, y, z) = ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 0,$$

be the equation to any conic: l, m, n the coordinates of any point. Let

$$A = bc - a'^2, \quad B = ca - b'^2, \quad C = ab - c'^2,$$

$$A' = aa' - b'c', \quad B' = bb' - c'a', \quad C' = cc' - a'b'.$$

Then the reciprocal equation to the conic is

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2A'\eta\zeta + 2B'\zeta\xi + 2C'\xi\eta = 0,$$

and in making $l\xi + m\eta + n\zeta = 0$, the ratios of ξ, η, ζ must be real if the tangents drawn from l, m, n are real: this will be found to imply that the determinant

$$\begin{vmatrix} A, & C', & B', & l \\ C', & B, & A', & m \\ B', & A', & C, & n \\ l, & m, & n, & 0 \end{vmatrix}$$

shall be negative*. This determinant may be shown† to be equal to the product of the determinant

$$\begin{vmatrix} a, & c', & b' \\ c', & b, & a' \\ b', & a', & c \end{vmatrix}$$

by the quantity

$$al^2 + bm^2 + cn^2 - 2a'mn - 2b'ln - 2c'lm,$$

that is, equal to $\phi(l, m, n) \times \square$.

Hence l, m, n is internal or external to $\phi(x, y, z)$ according as $\phi(l, m, n)$ and $\square\phi$ have the same or contrary sign.

If $\phi(l, m, n) = 0$, the point lies on the conic, and the point is neither internal nor external; if $\square\phi = 0$, the conic becomes a pair of straight lines, and no point can be said either to be within or without such a system. Hence our criterion fails, as it *ought to do*, just in the very two cases where the distinction vanishes. I believe that this criterion is here given for the first time.

* See theorem of the "Diminished Determinant" in Postscript to this paper.

† As we know *à priori* by virtue of a theorem given by M. Cauchy, and which is included as a particular case in a theorem of my own, relating to Compound Determinants, that is, Determinants of Determinants, which will take its place as an immediate consequence of my fundamental Theorem given in a Memoir about to appear. The well-known rule for the multiplication of Determinants is also a direct and simple consequence from my theorem on Compound Determinants, which indeed comprises, I believe, in one glance, all the heretofore existing Doctrine of Determinants.

To return to the two non-intersecting conics. Let us again throw them under the form

$$U = (x^2 + y^2) - e^2 (z^2 + y^2),$$
$$V = k (x^2 + y^2) - k\epsilon^2 (z^2 + y^2),$$

e and ϵ being real, that is, U and V being both functions corresponding to possible conics. Suppose U external to V; then *any point* in U is an external point to V.

Take in U either of the two points represented by the equations $y = 0$, $x^2 = e^2 z^2$; substituting these values of y and x, V becomes $k(e^2 - \epsilon^2) z^2$, and $\square V$ becomes $-k^3\epsilon^2 (1 - \epsilon^2)$; therefore $(1 - \epsilon^2)(e^2 - \epsilon^2)$ must be positive, that is, ϵ^2 must be one of the extremes of the three values 1, e^2, ϵ^2. In like manner, if V is external to U, e will be also one of the extremes of the same three quantities; and hence, if the two conics are mutually external, unity will be the middle magnitude of the group e^2, 1, ϵ^2.

Now the three roots of $\square (V + \lambda U) = 0$, are

$$\lambda = -k, \quad \lambda = -k\frac{\epsilon^2}{e^2}, \quad \lambda = -k\frac{1 - \epsilon^2}{1 - e^2}.$$

Hence if U and V be without one another, or, as it may be termed, are extra-spatial, the third value of λ will be of a different sign from the first two; but if the two conics be co-spatial, that is, if one includes the other, all the three values of λ will have the same sign. Hence we have the following elegant criterion of co-spatiality of two possible conics expressed by the equations $U = 0$, $V = 0$, between indeterminate coordinates ξ, η, ζ; the coefficients of the cubic function $\underset{\xi\eta\zeta}{\square}(\lambda U + \mu V)$ must give only changes or only continuations of sign.

If this test be not satisfied, it will remain to determine which of the two conics contains, and which is contained by the other. Let U contain V, then the order of magnitudes will be 1, e^2, ϵ^2; therefore $k\dfrac{1 - \epsilon^2}{1 - e^2}$ is greater than k, and therefore $k\dfrac{1 - \epsilon^2}{1 - e^2}$, which is that root of the equation $\square (V + \lambda U) = 0$ which is always one or the other of the extremes, is the *greatest* of the three. Hence the scheme for the impossible and possible regions of λ will be as below:

$-\infty$ ☜ Poss. A Imposs. B Poss. C Poss. ☞ $+\infty$

Hence if the two roots of $\dfrac{d}{d\lambda}\{V + \lambda U\} = 0$ be l and L, and of the two conics $V + lU = 0$, $V + LU = 0$, the former be the possible, and the latter the impossible one, U contains V or is contained in it according as l is greater or less than L.

Observe that if U and V be non-cospatial, so that the three values of μ in $\square(U+\mu V)=0$ have not all the same sign and consequently zero lies between the greatest and least of them, it will not be necessary to make trial of the characters of the two curves $U+mV=0$, and $U+MV=0$, in order to ascertain whether U and V intersect or not; for it will be sufficient to find which of the two quantities m and M substituted for μ in $\square(U+\mu V)$ causes it to have the opposite sign to $\square(U+0V)$, that is, $\square U$, and this one of the two it is, if either, which will make $U+\mu V$ an impossible conic, and will thus alone serve to determine whether the intersections of U and V are unreal, or the contrary.

It might be a curious question to consider whether, in a certain sense, conics not both possible may not be said to lie one within or without the other. Upon general logical grounds, I think it not improbable that two impossible conics might be discovered *each to contain the other;* but this is an inquiry which I have not had leisure to enter upon.

I have thus far supposed the roots of $\square(\lambda U+V)=0$ to be all distinct from one another. I now approach the discussion of the contact of two conics, in which event two or more of the roots will be equal. The condition for simple contact is evidently $\underset{\lambda\mu}{\square}\,\underset{\xi\eta\zeta}{\square}(\lambda U+\mu V)=0$.

The unpaired value of λ in $\square(\lambda U+V)$ makes $\lambda U+V$ an impossible pair of lines, and therefore, in the scheme for λ drawn as above, will separate the possible from the impossible region.

Whether the conics intersect in two real or two unreal points, besides the point of contact, will be known at once by ascertaining whether $U+\mu V=0$ represents two real or two imaginary lines. If the latter, the two curves lie *dos-à-dos* or one within the other, according as the successions of sign in $\square(\lambda U+V)$ are all of the same kind or not; if they be all of the same kind, one will include the other, namely, U will include V if the equal roots are greater, and be included in it if they be less than the unequal one. This last conclusion however, it should be observed, is inferred upon the principle of continuity, by making two values of λ approach indefinitely near to one another, but cannot be strictly deduced from the equations given for U and V applicable to the general case, in which the axes of coordinates are the three axes joining the vertices; since these latter, in the case supposed, reduce to two only, and consequently such representation of U and V becomes illusory.

If all three values of λ are equal, the three vertices come together, and hence the two conics will have three consecutive points in common, that is, will have the same circle of curvature. On this supposition the two curves cut at the point of contact, and all four points of intersection are of course real.

The classification of contacts between two conics may be stated as follows:

Simple contact = one case.

Second degree contact = two cases, namely, common curvature or double contact.

Third degree contact = one case, namely, contact in four consecutive points.

These four cases of course correspond to the several suppositions of there being two equal roots, three equal roots, two pairs of equal roots, or four equal roots in the biquadratic equation obtained between two variables by elimination performed in any manner between the given equations in the two conics.

The first species and the first case of the second species have been already disposed of. I proceed to assign the conditions appertaining to the second case of the second species, when U and V have a double contact.

Let A, A', B, B' be the two pairs of coincident points in which the conics are supposed to meet; either pair of lines AB, $A'B'$, and AB', $A'B$, becomes a coincident pair. Hence such a value of μ can be found as will make $U+\mu V$ the *square of a linear function of* ξ, η, ζ. If therefore we make $U+\mu V = W$, and form the determinant

$$\begin{vmatrix} \frac{d^2W}{d\xi^2}, & \frac{d^2W}{d\xi d\eta}, & \frac{d^2W}{d\xi d\zeta}, & p \\ \frac{d^2W}{d\eta d\xi}, & \frac{d^2W}{d\eta^2}, & \frac{d^2W}{d\eta d\zeta}, & q \\ \frac{d^2W}{d\zeta d\xi}, & \frac{d^2W}{d\zeta d\eta}, & \frac{d^2W}{d\zeta^2}, & r \\ p, & q, & r, & 0 \end{vmatrix}$$

$$= Ap^2 + Bq^2 + Cr^2 + 2Fqr + 2Grp + 2Hpq,$$

where all the coefficients are quadratic functions of μ, and make

$$A=0,\ B=0,\ C=0,\ F=0,\ G=0,\ H=0,$$

each of these six equations in μ will have one and the same root in common.

It is, however, enough to select any three; if these vanish together for any value of μ, the remaining three must also vanish. This is a simple application of a general law* which will appear in a forthcoming memoir on "Determinants and Quadratic Forms," of which this paper is to be considered as an accidental episode.

* For statement of this law called the Homaloidal Law, see *Philosophical Magazine* of this month "On Certain Additions, &c." [p. 150 below. ED.]

Take now any three of the six equations which for the sake of generality call $P=0$, $Q=0$, $R=0$. The hypothesis of double contact requires that P and Q, Q and R, R and P shall have a factor in common; but these conditions are not sufficiently explicit for our present object, since P, Q, R might be of the form

$$\kappa(\lambda-a)(\lambda-b), \quad \kappa'(\lambda-b)(\lambda-c), \quad \kappa''(\lambda-c)(\lambda-a),$$

and would thus satisfy the conditions above stated, without P, Q, R having a common factor. A sufficient criterion is that $fQ+gR$ and P shall have a common factor for all values of f and g.

Let then the resultant of $fQ+gR$ and P be

$$Lf^2+Mfg+Ng^2,$$

we must have

$$L=0, \quad M=0, \quad N=0,$$

where L is the resultant of P and Q,

N " " " R and Q;

and M is a new function, which if we call $Q=\phi(\lambda)$, $R=\psi(\lambda)$, and suppose a and b to be the two roots of $P=0$, is easily seen to be equal to

$$\phi a \,.\, \psi b+\phi b \,.\, \psi a.$$

This I call the connective of $P\,.\,Q$ and $P\,.\,R$.

L, M, N may conveniently be denoted by the forms

$$P\,.\,Q, \quad P\,.\,R, \quad Q\,.\,P\,.\,R.$$

We may now take more generally

$$aP+bQ+cR,$$
$$\alpha P+\beta Q+\gamma R,$$

which will have a factor in common for all values of a, b, c, α, β, γ.

I am indebted to Mr Cayley for the remark that the resultant of these two functions is a new quadratic function, which, according to my notation just given, may be put under the form

$$PQ(a\beta-b\alpha)^2+QR(b\gamma-c\beta)^2+RP(c\alpha-a\gamma)^2$$
$$+PRQ(b\gamma-c\beta)(c\alpha-a\gamma)+QPR(c\alpha-a\gamma)(a\beta-b\alpha)+RQP(a\beta-b\alpha)(b\gamma-c\beta).$$

Ternary systems of the six coefficients formed upon the type of (PQ, PQR, QR), I call *complete* systems, because the three functions included in such a system equated severally to zero, imply that the remaining three coefficients are all zero. Such a system as (PQ, QR, RP) I term an *incomplete* ternary system as not drawing with it the like implication. Probably (?) we should find on investigation that PRQ, QPR, RQP, would also be an incomplete system, but that systems formed after the type of PRQ, RQ, RQP are complete. This however is only matter of conjecture, as I have been too much occupied with other things to enter upon the inquiry. The distinct types of ternary systems are altogether six in number, namely, four of a symmetrical species,

$$PQ, \quad QR, \quad RP,$$

$$PRQ, \quad QPR, \quad RQP,$$

$$PQ, \quad PQR, \quad QR,$$

$$PRQ, \quad RQ, \quad RQP;$$

and two of an unsymmetrical species, namely,

$$PQ, \quad PQR, \quad PR,$$

$$PRQ, \quad RQ, \quad QPR.*$$

If instead of confining ourselves to three out of the six original quantities, $A, B, C; F, G, H$, we take them all into account, and write down the resultant of

$$aA + bB + cC + fF + gG + hH,$$

$$\alpha A + \beta B + \gamma C + \phi F + \chi G + \eta H;$$

we shall obtain a quadratic function of 15 variables (not however all independent) having 120 coefficients, all of which must be zero. It would be extremely interesting to determine how many *complete* ternary groups can be formed out of these 120 terms.

It will be recollected that we have assigned as the condition of contact in three consecutive points, that a certain cubic equation shall have all its roots real. Now, as well remarked by Mr Cayley, we cannot express this fact by less than three equations in integral terms of the coefficients. Thus if the cubic be written

$$a\lambda^3 + 3b\lambda^2 + 3c\lambda + d = 0,$$

we have as one of such ternary systems,

$$U = ac - b^2 = 0, \quad V = bd - c^2 = 0, \quad W = bc - ad = 0.$$

The significant parts of these equations are of course, however, capable of being connected by integral multipliers U', V', W', such that

$$U'U + V'V + W'W = 0.$$

* PQ, QR, RP, may be compared in a general way with the angles, and PRQ, QPR, RQP, with the sides of a triangle.

Any number of functions U, V, W so related, I call *syzygetic* functions, and U', V', W' I term the *syzygetic multipliers**. These in the case supposed are c, a, b, respectively.

In like manner it is evident that the members of any group of functions, more than two in number, whose nullity is implied in the relation of double contact, whether such group form a complete system or not, must be in syzygy.

Thus PQ, PQR, QR, must form a syzygy; nor is there any difficulty in assigning a system of multipliers to exhibit such syzygy. Calling $P=\phi(\lambda)$, $R=\psi(\lambda)$, a and b the two roots of $Q=0$, I have found that

$$\{(\psi a)^2+(\psi b)^2\}\,PQ-(\phi a\,.\,\psi a+\phi b\,.\,\psi b)\,PQR+\{(\phi a)^2+(\phi b)^2\}\,QR=0.$$

Again, if we take the *incomplete* system

$$(PQ),\quad (QR),\quad (RP),$$

it will be found that

$$L\,(QR)+M\,(RP)+N\,(PQ)=0,$$

provided that, calling a, b; c, d; e, f, the roots of $P=0$, $Q=0$, $R=0$, respectively, we make

$$L=(k_0+k_1a+k_2a^2+k_3a^3+k_4a^4)\frac{(a-c)(a-d)(a-e)(a-f)}{a-b}$$

$$+(k_0+k_1b+k_2b^2+k_3b^3+k_4b^4)\frac{(b-c)(b-d)(b-e)(b-f)}{b-a},$$

$$M=(k_0+k_1c+k_2c^2+k_3c^3+k_4c^4)\frac{(c-a)(c-b)(c-d)(c-e)(c-f)}{c-d}$$

$$+(k_0+k_1d+k_2d^2+k_3d^3+k_4d^4)\frac{(d-a)(d-b)(d-c)(d-e)(d-f)}{d-c},$$

$$N=(k_0+k_1e+k_2e^2+k_3e^3+k_4e^4)\frac{(e-a)(e-b)(e-c)(e-d)}{e-f}$$

$$+(k_0+k_1f+k_2f^2+k_3f^3+k_4f^4)\frac{(f-a)(f-b)(f-c)(f-d)}{f-e};$$

k_0, k_1, k_2, k_3, k_4 being quite arbitrary, and L, M, N, although presented in a fractional form, being essentially integral.

This fact of L, M, N constituting a system of multipliers to the syzygy QR, RP, PQ, is easily demonstrated; for

$$QR=(c-e)(c-f)(d-e)(d-f),$$

$$RP=(e-a)(e-b)(f-a)(f-b),$$

$$PQ=(a-c)(a-d)(b-c)(b-d).$$

* There will be in general various such systems of multipliers.

Hence $$L(QR) + M(RP) + N(PQ)$$

$$= (a-c)(a-d)(a-e)(a-f)(b-c)(b-d)(b-e)(b-f)(c-e)(c-f)(d-e)(d-f)$$

$$\times \Sigma \frac{k_0 + k_1 a + k_2 a^2 + k_3 a^3 + k_4 a^4}{(a-b)(a-c)(a-d)(a-e)(a-f)} = 0.$$

My theory of elimination enables me to explain exactly the nature of L, M, N, and the *reason* of their appearance as syzygetic factors.

Let L_r, M_r, N_r signify what L, M, N become, when all the k's except k_r are taken zero. Then the theory given by me in the *Philosophical Magazine* for the year 1838, or thereabouts†, shows that $L_0\lambda + L_1$ is the *prime derivee* of the first degree between the two equations P and $Q \times R$, or, in other words, will be the remainder integralized of $\frac{QR}{P}$.

In like manner $M_0\lambda + M_1$, $N_0\lambda + N_1$ are the integralized remainders of $\frac{RP}{Q}$ and of $\frac{PQ}{R}$ respectively.

If now the resultant of P, Q and of Q, R are each zero, but the resultant of P and R is not zero, it will be evident that P, Q, R must be of the form

$$f(\lambda + a)(\lambda + c), \quad g(\lambda + c)(\lambda + d), \quad h(\lambda + d)(\lambda + b);$$

and therefore $P \times R$ will contain Q, and consequently we must have

$$M_0 = 0, \quad M_1 = 0.$$

More generally, if we write

$$Q = 0,$$
$$\lambda Q = 0,$$
$$\lambda^2 Q = 0,$$
$$P \times R = 0,$$

and eliminate dialytically, that is, treating λ^4, λ^3, λ^2, λ as distinct quantities, we shall obtain*

$$\lambda^4 : \lambda^3 : \lambda^2 : \lambda : 1 :: M_4 : M_3 : M_2 : M_1 : M_0;$$

and therefore when $P \times R$ contains Q,

$$M_0 = 0, \quad M_1 = 0, \quad M_2 = 0, \quad M_3 = 0, \quad M_4 = 0.$$

* This cannot be obtained directly from what is stated in the paper referred to, although contained in the general theory of derivation there given. The arbitrary functions which enter into the expression for the general derivees have been in that paper evaluated only for the prime derivees, which however are only particular phenomena, with reference to the general results of Dialytic Elimination. Hereafter I may give a more general exposition of this remarkable, although ignored or neglected theory. The prime derivees of fx and $f'x$ are Sturm's Functions, cleared of quadratic factors, and are expressed by virtue of the general theorems there laid down as functions of x and of symmetrical functions of the roots of fx. [† p. 40 above. Ed.]

In like manner, when $Q \times P$ contains R,

$$N_0 = 0, \quad N_1 = 0, \quad N_2 = 0, \quad N_3 = 0, \quad N_4 = 0;$$

and when $R \times Q$ contains P,

$$L_0 = 0, \quad L_1 = 0, \quad L_2 = 0, \quad L_3 = 0, \quad L_4 = 0.$$

Accordingly, we see from the equation

$$L(QR) + M(RP) + N(PQ) = 0,$$

that if $QR = 0$, $RP = 0$; but PQ *not* $= 0$, then $N = 0$; and therefore

$$N_0 = 0, \quad N_1 = 0, \quad N_2 = 0, \quad N_3 = 0, \quad N_4 = 0,$$

and so in like manner for the remaining corresponding two suppositions*.

Before proceeding to consider the remaining case of the highest species of contact, I must observe that besides the equations involved in the condition that A, B, C; F, G, H, or, which is the same thing, that any three of them shall all have a factor in common, we must have $\square\,(U + \lambda V)$ containing the square of such common factor. In the memoir before adverted to a general theorem will be given and proved, which shows that this latter condition is involved in the former one; in fact, more generally (but still only as a particular case) that when U and V are quadratic functions of n letters, but $U + \epsilon V$ admits of being represented as a complete function of $(n-2)$ quantities only, which are themselves linear functions of the n letters, then $\square\,(U + \lambda V)$, which is of course a function of λ of the nth degree, will contain the factor $(\lambda - \epsilon)^2$.

When the two conics have four consecutive points in common, the characters of double-point contact and of contact in three consecutive points must exist simultaneously; and consequently the factor common to A, B, C; F, G, H, will enter not as a binary but as a ternary factor into $\square\,(U + \lambda V)$. This gives the extra condition required. As an example take the two conics,

$$U = \frac{y^2}{1-k} + x^2 - z^2 = 0,$$

$$V = y^2 + x^2 - 2kxz + (2k-1)\,z^2 = 0,$$

$$U + \lambda V = \left(\frac{1}{1-k} + \lambda\right) y^2 + (1+\lambda)\,x^2 - \{1 + \lambda\,(1-2k)\}\,z^2 - 2k\lambda xz.$$

* Since we are able to assign the values of the syzygetic multipliers in the equations

$$\begin{aligned}
L\ \ (PQ) + M\ \ (QR) + N\ \ (RP) &= 0,\\
L'\ (PQ) + M'\ (PQR) + N'\ (QR) &= 0,\\
L''\ (QR) + M''\ (QRP) + N''\ (RP) &= 0,\\
L'''\,(RP) + M'''\,(RPQ) + N'''\,(PQ) &= 0,
\end{aligned}$$

it follows that we may eliminate between these four equations any three of the six quantities (PQ), (PRQ), &c., and thus express any one of them in terms of any two others: this method, however, is not practically convenient. I may probably hereafter return to this subject.

The complete determinant of $U+\lambda V$ is then

$$\frac{-1}{1-k}\{1+(1-k)\lambda\}\{(1+\lambda)^2-2k\lambda(1+\lambda)+k^2\lambda^2\}=-\frac{1}{1-k}\{1+(1-k)\lambda\}^3.$$

A, B, C are the determinants of $U+\lambda V$, when $x=0$, $y=0$, $z=0$, respectively. Thus

$$A=\left(\frac{1}{1-k}+\lambda\right)(1+\lambda),$$

$$B=\left(\frac{1}{1-k}+\lambda\right)\{1+\lambda(1-2k)\},$$

$$C=k^2\lambda^2-(1+\lambda)\{1+\lambda(1-2k)\}=\lambda^2(1-k)^2-2\lambda(1-k)-1;$$

$\lambda=-\dfrac{1}{1-k}$ makes $A=0$, $B=0$, $C=0$, and the factor $\lambda+\dfrac{1}{1-k}$ enters cubed into $\square\,(U+\lambda V)$.

Hence the two conics have a contact of the third order.

This is easily verified; for if we pass from general to Cartesian and rectangular coordinates, and make z unity; $U=0$ will represent an ellipse with centre at the origin, eccentricity $\sqrt{k}$, and mean focal distance 1, and $V=0$ the circle of curvature at the extremity of the axis major*.

I had intended to have added some other remarks connected with the present discussion, and also to have appended an *à posteriori* proof of the propositions relative to the reality and otherwise of the vertices and chordal pairs of intersection which I have, at the commencement of this paper, deduced quite legitimately, but in a manner not at first sight perhaps easily intelligible, from the general principles of conjugate forms; but this discussion has run on already to a length so much greater than I had anticipated and than the importance of the inquiry may seem to justify, that I must reserve for a future number of the *Journal* what further matter I may have to communicate concerning it.

Postscript.—As I have alluded to Professor Boole's theorem relative to Linear Transformations, it may be proper to mention my theorem on the subject, which is of a much more general character, and includes Mr Boole's (so far as it refers to *Quadratic* Functions) as a corollary to a particular case. The demonstration will be given in the forthcoming memoir above alluded to.

Let U be a quadratic function of any number of letters $x_1, x_2 \dots x_n$, and let any number r of linear equations of the general form

$${}_1a_r x_1 + {}_2a_r x_2 + \dots\dots + {}_na_r x_n = 0,$$

* We have thus discussed all the four cases of biconical contact: for an exactly parallel discussion of the theory of contact of a plane with the curve of double curvature in which two surfaces of the second order intersect, see the paper in the *Philosophical Magazine* for this month, before referred to. [p. 148 below. Ed.]

be instituted between them: and by means of these equations let U be expressed as a function of any $(n-r)$ of the given letters, say of x_{r+1}, $x_{r+2} \ldots\ldots x_n$, and let U, so expressed, be called M. Let

$${}_1a_r x_1 + {}_2a_r x_2 + \ldots\ldots + {}_na_r x_n$$

be called L_r. Then the determinant of M in respect to the $(n-r)$ letters above given is equal to the determinant of

$$U + L_1 x_{n+1} + L_2 x_{n+2} + \ldots\ldots + L_r x_{n+r},$$

considered as a function of the $(n+r)$ letters

$$x_1 x_2 \ldots\ldots x_{n+r},$$

divided by the square of the determinant

$$\begin{vmatrix} {}_1a_1, & {}_2a_1 \ldots\ldots {}_ra_1 \\ {}_1a_2, & {}_2a_2 \ldots\ldots {}_ra_2 \\ \ldots & \ldots\ldots\ldots \\ {}_1a_r, & {}_2a_r \ldots\ldots {}_ra_r \end{vmatrix}.$$

This I call the theorem of Diminished Determinants.

If now we have U a function of r letters, and V of r other letters, and V is derived from U by linear transformations, that is, by r equations connecting the $2r$ letters; then, since U may be considered as a function of *all the* $2r$ letters with abortive coefficients for all the terms where any of the second set of r letters enter, we may apply our theorem of diminished determinants to the question so considered, and the result may be found to represent Mr Boole's theorem in a form rather more general and symmetrical, but substantially identical with that given by Mr Boole.

Thus suppose $\frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2$ say P, and $\frac{1}{2}\alpha u^2 + \beta uv + \frac{1}{2}\gamma v^2$ say Q, are mutually transformable by virtue of the linear equations

$$lx + my = \lambda u + \mu v,$$
$$l'x + m'y = \lambda' u + \mu' v,$$

P may be considered as a function of x, y, u, v, and Q as the value of P, when we eliminate x and y by virtue of the two linear equations

$$L_1 = lx + my - \lambda u - \mu v = 0,$$
$$L_2 = l'x + m'y - \lambda' u - \mu' v = 0;$$

we have therefore by our theorem the determinant of Q equal to the squared reciprocal of the determinant $\begin{vmatrix} l, & m \\ l', & m' \end{vmatrix}$, multiplied by the determinant

$$\begin{vmatrix} a, & b, & 0, & 0, & l, & l' \\ b, & c, & 0, & 0, & m, & m' \\ 0, & 0, & 0, & 0, & -\lambda, & -\lambda' \\ 0, & 0, & 0, & 0, & -\mu, & -\mu' \\ l, & m, & -\lambda, & -\mu, & 0, & 0 \\ l', & m', & -\lambda', & -\mu', & 0, & 0 \end{vmatrix},$$

which last determinant is evidently equal to the determinant of P multiplied by the square of the determinant $\begin{vmatrix} \lambda, & \mu \\ \lambda', & \mu' \end{vmatrix}$. Whence we see that the determinant of Q divided by the square of $\begin{vmatrix} \lambda, & \mu \\ \lambda', & \mu' \end{vmatrix}$, is equal to the determinant of P divided by the square of $\begin{vmatrix} l, & m \\ l', & m' \end{vmatrix}$. There is also another way more simple, but less direct, by means of which the theorem of diminished determinants may be made to yield Mr Boole's theorem of transformation*. Some unavowed use has been made in the foregoing pages of this former theorem, one of the highest importance in the analytical and geometrical theory of quadratic functions. It has been nearly a year in my possession, and I trust and believe that I am committing no act of involuntary misappropriation in announcing it as a result of my own researches.

* Namely, by considering P and Q as *each* derived from some common function of x, y, u, v, w, by means of the equations $L_1=0$, $L_2=0$; the law of Diminished Determinants will then indicate the determinants of P and Q, each under the form of fractions having the *same numerator*, but whose denominators will be $\begin{vmatrix} \lambda, & \mu \\ \lambda', & \mu' \end{vmatrix}^2$ and $\begin{vmatrix} l, & m \\ l', & m' \end{vmatrix}$ respectively.