ON THE CARDINAL POINTS OF AN OPTICAL INSTRUMENT.

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THE chief object of the following communication is to explain an elementary method of establishing the fundamental properties of the cardinal points of any optical instrument symmetrical about an axis. The method is also applied to determine the foci and focal lengths of a system of refractors, whose focal lengths and principal foci are given. The distinguishing feature of the investigation is that the relative positions of the refracting surfaces are defined by means of the distances between their principal foci instead of the distances between the surfaces themselves; some considerable simplification in the formulæ is thereby effected. To illustrate the method, the necessary formulæ for finding the cardinal points of an eye are set out at full length.

It is proposed, therefore, to establish the following known properties of an optical instrument.

If F and F' be the first and second principal foci, Q any point on the axis, Q' its image, m the magnification of a small object placed at Q, reckoned negative if the image is inverted,

$$\frac{1}{m} = -\frac{FQ}{f} = -\frac{f'}{F'Q'} \quad\text{.................(1)},$$

and therefore $$FQ \, . \, F'Q' = ff' \quad\text{.....................(2)},$$

where f and f' are the first and second focal lengths, and are such that

$$\frac{f}{\mu} = \frac{f'}{\mu'} \quad\text{.........................(3)},$$

where μ and μ' are the absolute refractive indices of the initial and final media. It is understood that lines measured from the first principal focus are to be regarded as positive, if measured opposite to the direction in which light passes through the instrument; and that lines from the second principal focus are positive, if measured in the opposite direction.

These properties are easily established for a single refracting surface. To establish them for any system of refracting surfaces it will therefore be sufficient to show that, if they hold for each of two refracting systems S_1 and S_2, they hold for the combination of the two.

Let F_1 and F_1' be the first and second principal foci of the first system S_1, f_1 and f_1' the focal lengths, μ and μ_1 the refractive indices of the first and last media of S_1. Let F_2 and F_2' be the foci, f_2 and f_2' the focal lengths of S_2, and let μ_1 and μ' be the indices of the first and last media of S_2.

Hence $$\frac{f_1}{\mu} = \frac{f_1'}{\mu_1} \text{ and } \frac{f_2}{\mu_1} = \frac{f_2'}{\mu'}.$$

Let $F_2F_1' = c$, so that c, being measured from a first principal focus, is positive if light travels in the direction from F_1' to F_2. Let Q be a point on the axis, q its image in S_1, Q' the image of q in S_2. Let m_1, m_2 be the successive linear magnifications of a small object at Q; m the total magnification, so that $m = m_1 m_2$. Then

$$\frac{1}{m_1} = -\frac{F_1Q}{f_1},\ \frac{1}{m_2} = -\frac{F_2q}{f_2}.$$

Therefore
$$\begin{aligned}\frac{1}{m} &= \frac{F_1Q}{f_1}\frac{F_2q}{f_2}\\ &= \frac{F_1Q}{f_1f_2}(c - F_1'q)\\ &= \frac{F_1Q}{f_1f_2}\left(c - \frac{f_1f_1'}{F_1Q}\right)\\ &= \frac{c}{f_1f_2}\left(F_1Q - \frac{f_1f_1'}{c}\right).\end{aligned}$$

Now take a point F on the axis, such that

$$F_1F = \frac{f_1f_1'}{c},$$

and write

$$f = -\frac{f_1f_2}{c};$$

then $$\frac{1}{m} = -\frac{FQ}{f}.$$

Similarly it may be proved that

$$m = -\frac{F'Q'}{f'},$$

where F' is a point on the axis, such that

$$F_2'F' = \frac{f_2 f_2'}{c},$$

and $$f' = -\frac{f_1' f_2'}{c}.$$

Hence Q' is determined by the relation

$$FQ.F'Q' = ff'.$$

Also $$\frac{f}{f'} = \frac{f_1 f_2}{f_1' f_2'} = \frac{\mu}{\mu_1}\frac{\mu_1}{\mu'} = \frac{\mu}{\mu'},$$

and therefore $$\frac{f}{\mu} = \frac{f'}{\mu'}.$$

Now F is conjugate to F_2 with respect to the system S_1, since $F_1F.F_1'F_2 = f_1 f_1'$. Hence rays from F in the first medium would pass through F_2 after being refracted by S_1, and would be parallel to the axis after passing through S_2. Hence F is the first principal focus as usually defined. Similarly F' is conjugate to F_1' with respect to S_2, and is therefore the second principal focus of the combination. It follows therefore, by induction, that the formulæ (1), (2), (3) must hold for any system of refracting surfaces, since they are known to hold for a single refractor.

The principal points H, H', or points of unit magnification, are determined by the equations

$$FH = -f, \quad F'H' = -f'.$$

It follows immediately, from Helmholtz's formula connecting the magnification at a point with the inclinations to the axis of a ray through the point before and after refraction (Heath's *Optics*, Art. 50), that the points N, N', determined by the equations

$$FN = -f', \quad F'N' = -f,$$

possess the property that any incident ray through N emerges through N' parallel to its original direction. These points N, N' are the nodal points.

The preceding results may be used to obtain the foci and focal lengths of any system of refractors, whose foci and focal lengths are given. Let F_1 and F_1', F_2 and F_2', ..., F_n and F_n' be the principal foci of the refractors; f_1 and f_1', ..., f_n and f_n' their focal lengths; and let $F_2F_1' = c_{12}$, $F_3F_2' = c_{23}$, ..., $F_nF_{n-1}' = c_{n-1,n}$, so that c_{12} is positive if the direction from F_2 to F_1' is opposite to the direction of light. Let F_{1r}, F_{1r}', f_{1r}, f_{1r}' be the foci and focal lengths of the combination of the refractors numbered 1, 2, 3, ..., r.

Then $$F_2'F_{12}' = \frac{f_2f_2'}{c_{12}};$$

$$F_3'F_{13}' = \frac{f_3f_3'}{F_3F_{12}'} = \frac{f_3f_3'}{c_{23} - F_2'F_{12}'}$$

$$= \frac{f_3f_3'}{c_{23} -} \frac{f_2f_2'}{c_{12}};$$

$$F_4'F_{14}' = \frac{f_4f_4'}{F_4F_{13}'} = \frac{f_4f_4'}{c_{34} - F_3'F_{13}'}$$

$$= \frac{f_4f_4'}{c_{34} -} \frac{f_3f_3'}{c_{23} -} \frac{f_2f_2'}{c_{12}};$$

and generally

$$F_n'F_{1n}' = \frac{f_nf_n'}{c_{n-1,n} -} \frac{f_{n-1}f'_{n-1}}{c_{n-2,n-1} -} \cdots - \frac{f_2f_2'}{c_{12}}.$$

Using the principle of the reversibility of a ray of light, we can now write down the formula to find F_{1n}, viz.,

$$F_1F_{1n} = \frac{f_1f_1'}{c_{12} -} \frac{f_2f_2'}{c_{23} -} \frac{f_3f_3'}{c_{34} - \cdots -} \frac{f_{n-1}f'_{n-1}}{c_{n-1,n}}.$$

The foci of the combination of n refractors are therefore determined. The focal lengths are determined by the following equations:—

$$f_{12} = -\frac{f_1f_2}{F_2F_1'}, \quad f_{13} = -\frac{f_3f_{12}}{F_3F_{12}'},$$

$$f_{14} = -\frac{f_4 f_{13}}{F_4 F_{13}'}, \quad \ldots\ldots\ldots\ldots\ldots\ldots$$

$$f_{1n} = -\frac{f_n f_{1,n-1}}{F_n F_{1,n-1}};$$

whence $$f_{1n} = (-1)^{n-1} \frac{f_1 f_2 f_3 \ldots f_n}{F_2 F_1' . F_3 F_{12}' . F_4 F_{13}' \ldots F_n F_{1,n-1}'}.$$

Or the focal lengths may be expressed in terms of the given quantities in the following way. From the above equations

$$-\frac{f_1 f_2}{f_{12}} = c_{12};$$

$$\frac{f_3}{f_{13}} = -\frac{1}{f_{12}} F_3 F_{12}' = -\frac{1}{f_{12}} (c_{23} - F_2' F_{12}')$$

$$= -\frac{1}{f_{12}} \left(c_{23} - \frac{f_2 f_2'}{c_{12}}\right),$$

and therefore

$$\frac{f_1 f_2 f_3}{f_{13}} = c_{12} c_{23} - f_2 f_2'.$$

Again, $$-\frac{f_4}{f_{14}} = \frac{1}{f_{13}} (c_{34} - F_3' F_{13}')$$

$$= \frac{1}{f_{13}} \left(c_{34} - \frac{f_3 f_3'}{F_3 F_{12}'}\right)$$

$$= \frac{1}{f_{13}} \left(c_{34} + f_3 f_3' \frac{f_{13}}{f_3 f_{12}}\right);$$

whence $$(-1)^3 \frac{f_1 f_2 f_3 f_4}{f_{14}} = c_{34} \frac{(-1)^2 f_1 f_2 f_3}{f_{13}} - f_3 f_3' \frac{(-1) f_1 f_2}{f_{12}}.$$

And generally

$$(-1)^{n-1} \frac{f_1 f_2 \ldots f_n}{f_{1,n}} = c_{n-1,n} \frac{(-1)^{n-2} f_1 f_2 \ldots f_{n-1}}{f_{1,n-1}}$$

$$- f_{n-1} f_{n-1}' \frac{(-1)^{n-3} f_1 f_2 \ldots f_{n-2}}{f_{1,n-2}}.$$

It follows from these equations that

$$\frac{(-1)^{n-1} f_1 f_2 \cdots f_n}{f_{1,n}}$$

is equal to the numerator of the last convergent to the continued fraction,

$$c_{12} - \frac{f_2 f_2'}{c_{23} -} \, \frac{f_3 f_3'}{c_{34} -} \cdots - \frac{f_{n-1} f_{n-1}'}{c_{n-1\,n}}.$$

If therefore K denotes this last numerator,

$$f_{1n} = (-1)^{n-1} \frac{f_1 f_2 \cdots f_n}{K} \quad \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

Similarly,

$$f_{1n}' = (-1)^{n-1} \frac{f_1' f_2' \cdots f_n'}{K'} \quad \ldots\ldots\ldots\ldots (A'),$$

where K' is the numerator of the last convergent to

$$c_{n-1,n} - \frac{f_{n-1} f'_{n-1}}{c_{n-2,n-1} -} \cdots - \frac{f_2 f_2'}{c_{12}}.$$

Now it has been proved that, if μ and μ_n are the indices of the first and last media,

$$\frac{f_{1n}}{f_{1n}'} = \frac{\mu}{\mu_n} = \frac{\mu}{\mu_1} \frac{\mu_1}{\mu_2} \cdots \frac{\mu_{n-1}}{\mu_n} = \frac{f_1 f_2 \cdots f_n}{f_1' f_2' \cdots f_n'}.$$

It follows therefore that $K = K'$, an equation which leads to a known property of continued fractions.

It is easily deduced that the formulæ which determine the positions of F_{1n} and F_{1n}' may be written

$$F_1 F_{1n} = \frac{f_1 f_1'}{K} \frac{\partial K}{\partial c_{12}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (B),$$

$$F_n' F_{1n}' = \frac{f_n f_n'}{K} \frac{\partial K}{\partial c_{n-1,n}} \ldots\ldots\ldots\ldots\ldots (B').$$

The equations (A), (A'), (B), (B'), with the relation $f_{1n}/\mu = f_{1n}'/\mu_n$, contain all the necessary information.

For example, in the case of four refracting surfaces,

$$K = c_{12} c_{23} c_{34} - c_{12} f_3 f_3' - c_{34} f_2 f_2',$$

$$f_{14} = -\frac{f_1 f_2 f_3 f_4}{K}, \quad f_{14}' = -\frac{f_1' f_2' f_3' f_4'}{K} = \frac{\mu_4}{\mu} f_{14},$$

$$F_1 F_{14} = \frac{f_1 f_1'}{K}(c_{23} c_{34} - f_3 f_3'), \quad F_4' F_{14}' = \frac{f_4 f_4'}{K}(c_{12} c_{23} - f_2 f_2').$$

It will be found that frequently the formulæ obtained here will be more convenient for numerical calculation than those usually given in treatises on Optics.

Thus, to find the cardinal points for an eye we may proceed as follows. Let r_1, r_2, r_3 be the numerical values of the radii of curvature of the cornea, and the anterior and posterior surfaces of the crystalline lens. Let μ, μ_1, μ_2, μ_3 be the refractive indices of air, the aqueous humour, crystalline lens and vitreous humour. Let a be the distance between the anterior surfaces of the cornea and lens, b the thickness of the lens.

Then

$$f_1 = \frac{\mu r_1}{\mu_1 - \mu}, \quad f_1' = \frac{\mu_1 r_1}{\mu_1 - \mu} = f_1 + r_1,$$

$$f_2 = \frac{\mu_1 r_2}{\mu_2 - \mu_1}, \quad f_2' = \frac{\mu_2 r_2}{\mu_2 - \mu_1} = f_2 + r_2,$$

$$f_3 = \frac{\mu_2 r_3}{\mu_2 - \mu_3}, \quad f_3' = \frac{\mu_3 r_3}{\mu_2 - \mu_3} = f_3 - r_3.$$

Write $c_1 \; (= -c_{12}) = f_1' + f_2 - a,$

$$c_2 \; (= -c_{23}) = f_2' + f_3 - b,$$

$$K = c_1 c_2 - f_2 f_2'.$$

Then the first principal focus is in front of the cornea at a distance from its anterior surface equal to

$$f_1 - \frac{f_1 f_1' c_2}{K}.$$

The second principal focus is behind the crystalline lens at a distance from its posterior surface equal to

$$f_3' - \frac{f_3 f_3' c_1}{K}.$$

The first and second focal lengths are given by

$$f = \frac{f_1 f_2 f_3}{K}, \quad f' = \frac{f_1' f_2' f_3'}{K} = \frac{\mu_3}{\mu} f.$$

The first principal point and the first nodal point are at distances f and f' respectively behind the first principal focus. The second principal point and the second nodal point are at distances f' and f respectively in front of the second principal focus.

The formulæ have here been adapted so that each letter shall represent a positive quantity in the case of the normal human eye.

NOTE ON KIRKMAN'S PROBLEM.

By *A. C. Dixon, M.A.*

I do not know whether it has been remarked that the solutions of Kirkman's Problem may be divided into two classes as follows:

Suppose one of the school girls to receive an apple, another an orange, another a pear, and another a plum, and each of the others two, three or four of these fruits, no two receiving alike and none receiving two of a kind. Then it is possible for thirty-five triads to be formed, each of which will have an even number of each kind of fruit, and the triads may be broken up into seven sets of five each including all the girls.

Let us denote the girls by $a, b, c, d, e, f, g, h, i, j, k, l, m, n, o$ or $\alpha, \beta, \alpha\beta, \gamma, \delta, \gamma\delta, \alpha\gamma, \alpha\beta\gamma\delta, \beta\delta, \alpha\delta, \beta\gamma\delta, \alpha\beta\gamma, \alpha\gamma\delta, \alpha\beta\delta, \beta\gamma$. Then the following is such an arrangement—

abc . *adg* . *aej* . *afm* . *ahk* . *ain* . *alo*,

def . *bhm* . *bdo* . *bgl* . *bjn* . *bfk* . *bei*,

ghi . *cij* . *cfh* . *cen* . *cdl* . *cgo* . *ckm*,

jkl . *eko* . *gkn* . *dik* . *egm* . *djm* . *dhn*,

mno . *fln* . *ilm* . *hjo* . *fio* . *ehl* . *fgj*.

In each triad if the second notation is used there will be an even number (2 or 0) of each of the symbols $\alpha, \beta, \gamma, \delta$.

In this arrangement let us take any two triads containing the same letter, as *alo*, *fio*. Then if a, f, i, l are taken in pairs