NUMBER OF PROPER QUATERNARY n-ICS.

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's Coll., London.

1. *Introduction*. A QUANTIC of n^{th} degree in x, y, z may be said to be *complete* when all the possible terms are present, and to be *incomplete* when some of the possible terms are absent, so that it contains a number of terms r less than the full number of terms of the complete quantic.

A quantic of n^{th} degree may be said to be a *proper* quantic when it is not the product of algebraic factors of lower degrees, and is not homogeneous. A quantic which contains a linear or other algebraic factor, or is homogeneous, may be styled an *improper* quantic.

It is proposed to investigate in this Paper the number of *incomplete proper quaternary* n^{th} degree quantics, arising from the complete n^{th} degree quaternary quantic by erasure of some of its terms. This number is interesting as being also the number of proper n-ic equations.

[This Paper is an extension to quaternary quantics of the author's Paper on the "Number of Proper Ternary Quantics," published at p. 1 of this volume. The procedure and notation will be as far as possible the same as, or similar to, the procedure and notation in that Paper. The numbering of the Articles and Results of that Paper will also be followed as closely as possible; though of course the Results in this Paper are far more numerous. In order to make this Paper complete in itself a good deal of repetition has been unavoidable.]

From the definitions it follows that every complete quaternary quantic is (in general) a *proper* quaternary quantic.

2. *Preliminary Formulæ*. Let n denote the degree of a function of x, y, z.

Let $u^{(n)}{}_n$ denote a *binary quantic* of n^{th} degree, so that in general

$$u^{(n)}{}_n = a^{(n)}{}_{n,0}\, x^n + a^{(n)}{}_{n-1,1}\, x^{n-1}y + \dots$$
$$\dots\dots\dots\dots + a^{(n)}{}_{1,n-1}\, xy^{n-1} + a_{0,n}\, y^n \;\dots\dots(1).$$

Let U_0 denote a *ternary quantic* of n^{th} degree, so that in general

$$U_0 = u^{(0)}{}_0, \text{ (a constant)} \;\dots\dots\dots\dots(2a),$$

$$U_1 = (a'_{1,0}\, x + a'_{0,1}\, y + a'_{0,0}\, z = u'_1 + u'_0 . z \;\dots\dots\dots\dots(2b),$$

$$U_2 = (a''_{2,0}x^2 + a''_{1,1}xy + a''_{0,2}y^2) + (a''_{1,0}x + a''_{0,1}y)z + a''_{0,0}z^2$$
$$= u''_2 + u''_1.z + u''_0.z^2 \text{.........(2c)},$$

$$U_3 = (a'''_{3,0}x^3 + a'''_{2,1}x^2y + a'''_{1,2}xy^2 + a'''_{0,3}y^3)$$
$$+ (a'''_{2,0}x^2 + a'''_{1,1}xy + a'''_{0,2}y^2)z + (a'''_{1,0}x + a'''_{0,1}y)z^2 + a'''_{0,0}z^3$$
$$= u'''_3 + u'''_2.z + u'''_1.z^2 + u'''_0.z^3 \text{.........(2d)},$$
$\vdots$

and, in general,

$$U_n = u^{(n)}_n + u^{(n)}_{n-1}.z + u^{(n)}_{n-2}.z^2 + \text{.........}$$
$$u^{(n)}_1.z^{n-1} + u^{(n)}.z^n \text{.........(2e)}.$$

Lastly let Υ_n denote a quaternary quantic of n^{th} degree in *three* variables (x, y, z), so that, in general

$$\Upsilon_n = U_n + U_{n-1} + \text{......} + U_2 + U_1 + U_0 \text{ ....(3)}.$$

Next let t_n, T_n, τ_n denote the number of terms in $u^{(n)}_n$, U_n, Υ_n respectively; therefore

$$t_n = (n+1) \text{ ....(4a)},$$
$$T_n = \Sigma^n_0(t_n) = \{(n+1) + n + \ldots + 3 + 2 + 1\}$$
$$= \tfrac{1}{2}(n+1)(n+2)\} \text{ ....(4b)},$$
$$\tau_n = \Sigma^n_0(T_n) = \Sigma^n_0\{\tfrac{1}{2}(n+1)(n+2)\}$$
$$= \tfrac{1}{6}(n+1)(n+2)(n+3) \text{ ......(4c)}.$$

Let $C(n, r)$ denote the number of combinations of n different taken r together, so that

$$C(n, r) = n!/r!(n-r)! \text{ ....(5)}.$$

Let $S(n, r)$ = number of sets, containing r terms, formable from Υ_n.

Let $s(n, r)$ = number of sets, containing r terms, formable from Υ_n, each set containing at least one term from U_n.

Let $\Sigma(n, r)$ = number of functions of n^{th} degree, containing r terms, (including improper quantics) formable from Υ_n.

Let $\sigma(n, r)$ = number of *improper* quantics of n^{th} degree, containing r terms, formable from Υ_n.

Let $N'''(n, r)$, $N^{iv}(n, r)$ be the number of proper *ternary* and *quaternary* quantics of n^{th} degree, containing r terms, formable from U_n, Υ_n respectively.

Let $N'''(n)$, $N^{iv}(n)$ be the total number of proper *ternary* and *quaternary* quantics of n^{th} degree, formable from U_n, Υ_n respectively.

Thus $N^{iv}(n, r)$, $N^{iv}(n)$ are the numbers now sought, whilst $N'''(n, r)$, $N'''(n)$ are those investigated in the previous Paper (p. 2).

Then

$$S(n, r) = C(\tau_n, r) = \tau_n!/r!\,(n-r)! \quad \text{......(6)},$$

$$s\,(n, r) = S(n, r) - S\{(n-1), r\} \quad \text{..........(7)},$$

$$= C(\tau_n, r) - C(\tau_{n-1}, r) \quad \text{............(7a)},$$

$$\Sigma\,(n, r) = s\,(n, r) \quad \text{..................................(8)},$$

$$N^{iv}(n, r) = \Sigma\,(n, r) - \sigma\,(n, r) \quad \text{.................(9)},$$

$$= s\,(n, r) - \sigma\,(n, r) \quad \text{...............(9a)}.$$

The computation of $\sigma\,(n, r)$ will occupy most of the rest of this Paper; and, in fact, presents the only difficulty.

3. *Decomposition of $\sigma\ (n,\ r)$ into parts.* A quaternary n-ic is an *improper* quantic in following cases:

I. When containing one, or more, of x, y, z as factors.

II. When it is a function of *only one* of the variables x, y, z.

III. When it is a homogeneous function of *only two* of the variables x, y, z.

IV. When it is a non-homogeneous function of *only two* of the variables x, y, z.

V. When it is a homogeneous function of the three variables.

These cases are to a considerable extent mutually involved, *e.g.*

$(ax^n + bx^m + cx^p)$ falls under both Cases I., II.

$(ax^n + bx^\mu y^m + cx^\varpi y^p)$ falls under both Cases I., IV.

$(ax^n + bx^{n-m} y^m + cx^{n-\varpi-p} y^\varpi z^p)$ falls under both Cases I., V., &c., &c., &c.

and the only difficulty consists in avoiding counting such cases more than once. A special symbol will now be defined to denote the *number* of functions falling under each Case I. to V., *such as to exclude all falling under previous Cases.*

I. Let $\sigma\,(n, r;\ x, y, z;\ yz, zx, xy;\ xyz)$ be the number of n-ic functions of r terms, containing *one or more* of x, y, z as factors.

II. Let $\sigma(n, r; fx, fy, fz)$ be the number of n-ic functions of r terms which are functions of only one variable, but not containing x, y, or z as factors.

III. Let $\sigma(n, r; f(y:z), f(z:x), f(x:y)$ be the number of *homogeneous binary* n-ic functions, (*i.e.* homogeneous n-ic functions of two variables only), of r terms, not containing x, y, or z as factors.

IV. Let $\sigma\{n, r; f(y, z), f(z, x), f(x, y)\}$ be the number of *non-homogeneous ternary* n-ic functions of *two* variables containing r terms, but not containing x, y, or z as factors; (these are clearly *proper ternary* n-ics of r terms.)

V. Let $\sigma\{n, r; f(x:y:z)\}$ be the number of *homogeneous ternary* n-ic functions of *three* variables, containing r terms, but not containing x, y, or z as factors.

Collecting the five parts of $\sigma(n, r)$, it follows that

$$\sigma(n,r) = \sigma(n,r;\ x,y,z;\ yz,zx,xy;\ xyz)$$
$$+\,\sigma(n,r;\ fx,fy,fz) + \sigma\{n,r;\ f(y:z), f(z:x), f(x:y)\}$$
$$+\,\sigma\{n,r;\ f(y,z), f(z,x), f(x,y)\} + \sigma\{n,r, f(x:y:z)\} \ldots (10).$$

It will be seen that the five parts of $\sigma(n, r)$ have been so defined as to exclude twice counting of functions falling under more than one of the five Cases I.—V.

4. *Number of terms containing x, y, z.*

Let ξ_n, X_n, $\mathfrak{X}_n$ be the number of terms containing x in $u^{(n)}{}_n$, U_n, Υ_n,

Let η_n, Y_n, $\mathfrak{Y}_n$ be the number of terms containing y in $u^{(n)}{}_n$, U_n, Υ_n,

Let ζ_n, Z_n, $\mathfrak{Z}_n$ be the number of terms containing z in $u^{(n)}{}_n$, U_n, Υ_n.

Let λ_n, L_n, $\mathfrak{L}_n$ be the number of terms containing both y, z in $u^{(n)}{}_n$, U_n, Υ_n,

Let μ_n, M_n, $\mathfrak{M}_n$ be the number of terms containing both z, x in $u^{(n)}{}_n$, U_n, Υ_n,

Let ν_n, N_n, $\mathfrak{N}_n$ be the number of terms containing both x, y in $u^{(n)}{}_n$, U_n, Υ_n,

Let ϖ_n, P_n, $\mathfrak{P}_n$ be the number of terms containing xyz in $u^{(n)}{}_n$, U_n, Υ_n.

Then, recurring to the definitions of $u^{(n)}{}_n$, U_n, Υ_n in equations (1), (2e), (3), it follows that

$$\xi_n = \eta_n = \zeta_n \quad = n \text{ ....(11)},$$

$$X_n = Y_n = Z_n \quad = \{n + (n-1) + \ldots + 3 + 2 + 1\}$$

$$= \tfrac{1}{2}n(n+1) = T_{n-1} \text{ ....(11}a\text{)},$$

$$\mathfrak{X}_n = \mathfrak{Y}_n = \mathfrak{Z}_n \quad = \Sigma^n{}_0 (X_n) = \Sigma^n{}_0 \tfrac{1}{2}n(n+1)$$

$$= \tfrac{1}{6}n(n+1)(n+2) = \tau_{n-1} \text{ ......(11}b\text{)},$$

$$\lambda_n = \mu_n = \nu_n \quad = (n-1) \text{ ....(12)},$$

$$L_n = M_n = N_n \quad = \{(n-1) + (n-2) + \ldots + 3 + 2 + 1\}$$

$$= \tfrac{1}{2}n(n-1) = T_{n-2} \text{ ....(12}a\text{)},$$

$$\mathfrak{L}_n = \mathfrak{M}_n = \mathfrak{N}_n = \Sigma^n{}_1 (L_n) = \Sigma^n{}_1 \tfrac{1}{2}n(n-1)$$

$$= \tfrac{1}{6}n(n-1)(n-2) = \tau_{n-2} \text{ ....(12}b\text{)},$$

Lastly $\varpi = 0$ ....(13),

and, observing the form of U_n in equation (2e), it is seen that the three terms $u^{(n)}{}_n$, $u^{(n)}{}_{n-1}.z^{n-1}$, $u^{(n)}{}_0.z^n$ contribute nothing to P_n, whilst each intermediate term, such as $u^{(n)}{}_{n-q}.z^q$ contributes its quota ν_q (see notation above) to P_n,

therefore $P_n = \{(\nu - 1) + (\nu - 2) + \ldots + 3 + 2 + 1\}$

$$= \tfrac{1}{2}(n-1)(n-2) = T_{n-3} \text{ ....(13}a\text{)},$$

$$\mathfrak{P}_n = \Sigma^n{}_1 P_n = \Sigma^n{}_1 \tfrac{1}{2}(n-1)(n-2)$$

$$= \tfrac{1}{6}n(n-1)(n-2) = \tau_{n-3} \text{ ......(13}b\text{)},$$

5. *Decomposition of* $\sigma(n, r;\ x, y, z;\ yz, zx, xy;\ xyz)$. This may be decomposed into the algebraic sum of seven parts, viz.

3 parts, when only one of x, y, z enter as factors.

3 parts, when only two of x, y, z enter as factors.

1 part, when x, y, and z all enter as factors.

Special symbols will now be defined for these.

Let $S(n,r,x)$, $S(n,r,y)$, $S(n,r,z)$; $S(n,r,yz)$, $S(n,r,zx)$, $S(n,r,xy)$; $S(n,r,xyz)$ be the number of sets of r terms containing either x, or y, or z; or yz, or zx, or xy; or xyz as a common factor formable from the $\mathfrak{X}_n$, $\mathfrak{Y}_n$, $\mathfrak{Z}_n$; $\mathfrak{L}_n$, $\mathfrak{M}_n$, $\mathfrak{N}_n$; or $\mathfrak{P}_n$ terms of Υ_n which by definition (Art. 4) contain these factors respectively.

Let $s(n, r, x)$, $s(n, r, y)$, $s(n, r, z)$; $s(n, r, yz)$, $s(n, r, zx)$; $s(n, r, xy)$; $s(n, r, xyz)$ be the number of sets formed precisely like the preceding S, except that each set is to contain at least one term from U_n.

Let $\sigma(n, r, x)$, $\sigma(n, r, y)$, $\sigma(n, r, z)$; $\sigma(n, r, yz)$, $\sigma(n, r, zx)$, $\sigma(n, r, xy)$; $\sigma(n, r, xyz)$ be the number of n-ic functions of r terms, containing either x, or y, or z; or yz, or zx, or xy; or xyz respectively as a common factor, formable from Υ_n.

Computing from above definitions:—

$$S(n, r, x) = S(n, r, y) = S(n, r, z) = C(\mathfrak{X}_n, r)$$
$$= C(\tau_{n-1}, r) \ldots\ldots\ldots (14),$$

$$s(n, r, x) = s(n, r, y) = s(n, r, z) = S(n, r, x) - S(n-1, r, x) \ldots,$$

$$\sigma(n, r, x) = \sigma(n, r, y) = \sigma(n, r, z) = s(n, r, x)$$
$$= S(n, r, x) - S(n-1, r, x)$$
$$= C(\tau_{n-1}, r) - C(\tau_{n-2}, r) \ \ldots\ldots (15).$$

Again, $S(n, r, yz) = S(n, r, zx) = S(n, r, xy)$
$$= C(\mathfrak{L}_n, r) = C(\tau_{n-2}, r) \ldots\ldots\ldots (16),$$

$$s(n, r, yz) = s(n, r, zx) = s(n, r, xy)$$
$$= S(n, r, yz) - S(n-1, r, yz) \ldots,$$

$$\sigma(n, r, yz) = \sigma(n, r, zx) = \sigma(n, r, xy) = s(n, r, yz)$$
$$= S(n, r, yz) - S(n-1, r, yz)$$
$$= C(\tau_{n-2}, r) - C(\tau_{n-3}, r) \ \ldots\ldots (17).$$

Lastly, $S(n, r, xyz) = C(\mathfrak{P}_n, r) = C(\tau_{n-3}, r) \ \ldots\ldots (18),$

$$s(n, r, xyz) = S(n, r, xyz) - S(n-1, r, xyz) \ldots,$$

$$\sigma(n, r, xyz) = s(n, r, xyz) = S(n, r, xyz) - S(n-1, r, xyz)$$
$$= C(\tau_{n-3}, r) - C(\tau_{n-4}, r) \ldots (19).$$

Now, the *seven* numberings of type σ just computed are not mutually exclusive, but would—unless properly combined—include many repetitions, thus

$\sigma(n, r, x)$ includes all sets in $\sigma(n, r, zx)$, $\sigma(n, r, xy)$, $\sigma(n, r, xyz)$,

$\sigma(n, r, y)$ includes all sets in $\sigma(n, r, yz)$, $\sigma(n, r, xy)$, $\sigma(n, r, xyz)$,

$\sigma(n, r, z)$ includes all sets in $\sigma(n, r, yz)$, $\sigma(n, r, zx)$, $\sigma(n, r, xyz)$,

$\sigma(n, r, yz)$, $\sigma(n, r, zx)$, $\sigma(n, r, xy)$ each include all sets in $\sigma(n, r, xyz)$.

Combining these seven parts in such a way as to avoid twice counting of any sets, the final value is

$$\sigma(n, r, x, y, z;\ yz, zx, xy;\ xyz)$$
$$= \{\sigma(n, r, x) + \sigma(n, r, y) + \sigma(n, r, z)\}$$
$$- \{\sigma(n. r, yz) + \sigma(n, r, zx) + \sigma(n, r, xy)\}$$
$$+ \sigma(n, r, xyz) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$$
$$= 3C(\tau_{n-1}, r) - 6C(\tau_{n-2}, r) + 4C(\tau_{n-3}, r) - C(\tau_{n-4}, r) \ldots(21).$$

6. *Number of terms containing only one variable.*

Let $\mathfrak{X}'_n$, $\mathfrak{Y}'_n$, $\mathfrak{Z}'_n$ be the number of terms of form $a^{(m)}_{m,0}x^m$, or $a^{(m)}_{0,m}y^m$, or $u^{(m)}_0 z^m$ contained in Υ_n, where m takes all the values, 1, 2, 3, ..., n, (but not zero). Then on comparing the form of Υ_n in equation (3) with those of U_1, U_2, ..., U_n in equations (2*b*), ..., (2*e*), it is seen that each function U_m contributes *one* term of required form to each of the numbers $\mathfrak{X}_n'$, $\mathfrak{Y}_n'$, $\mathfrak{Z}_n'$,

therefore $\mathfrak{X}_n' = \mathfrak{Y}_n' = \mathfrak{Z}_n' = n$ ....(22).

7. *Computation of* $\sigma(n, r, fx, fy, fz)$. This may be decomposed into the sum of three parts as follows:

Let $\sigma(n, r, fx)$, $\sigma(n, r, fy)$, $\sigma(n, r, fz)$ be the number of n-ic functions, of r terms each, formable from Υ_n, which are functions of x only, or of y only, or of z only, and yet not containing x, y, or z as a factor.

It is clear that the functions included in the numbering $\sigma(n, r, fx)$, $\sigma(n, r, fy)$, $\sigma(n, r, fz)$ must each contain the constant term $u^{(0)}_0$, and also either $a^{(n)}_{n,0}x^n$, or $a^{(n)}_{0,n}y^n$, or $u^{(n)}_0 z^n$, and also $(r-2)$ *other* terms taken out of the above $\mathfrak{X}_n'$, $\mathfrak{Y}_n'$, $\mathfrak{Z}_n'$ terms of (Art. 6) respectively,

$$\therefore \sigma(n, r, fx) = \sigma(n, r, fy) = \sigma(n, r, fz) = C(\mathfrak{X}_m' - 1, r-2)$$
$$= C(n-1, r-2) \quad \ldots\ldots\ldots(23).$$

Hence, as the sets included in these three numbers σ are wholly different, the three parts σ are additive,

$$\therefore \sigma(n, r, fx, fy, fz) = \sigma(n, r, fx) + \sigma(n, r, fy) + \sigma(n, r, fz)$$
$$= 3C(n-1, r-2) \ldots\ldots\ldots\ldots\ldots\ldots(23a).$$

8. *Computation of* $\sigma\{n, r, f(y:z), f(z:x), f(x:y)\}$. This may be decomposed into the sum of three parts as follows:

Let $\sigma\{n, r, f(y:z)\}$, $\sigma\{n, r, f(z:x)\}$, $\sigma\{n, r, f(x:y)\}$ be the number of *homogeneous binary* n-ic functions of r terms, formable from Υ_n, which are homogeneous in y and z, or in z and x, or in x, y, and yet not containing any of x, y or z as a factor.

Referring to the form of U_n, Υ_n in equations (2e) and (3), it is seen that these functions must be all homogeneous binary n-ics contained in U_n, and must all contain the pair of terms,

$$(a^{(n)}{}_{0,n}y^n + u^{(n)}{}_0 . z^n), \text{ or } (u^{(n)}{}_0 . z^n + a^{(n)}{}_{n,0}x^n), \text{ or } (a^{(n)}{}_{n,0}x^n + a^{(n)}{}_{0,n}y_n)$$

respectively, and also $(r-2)$ other terms taken from the remaining $(n-1)$ terms of the complete binary n-ics in $y:z$, $z:x$, $x:y$ respectively contained in U_n; whereof, $u^{(n)}{}_n$ of equation (1) is the type-function of $x:y$,

$$\therefore \sigma\{n, r, f(y:z)\} = \sigma\{n, r, f(z:x)\} = \sigma\{n, r, f(x:y)\}$$
$$= C(n-1, r-2) \ldots\ldots\ldots\ldots\ldots (24).$$

Hence, as the sets included in the three numbers σ above are wholly different, these three parts are additive, and

$$\sigma\{n, r, f(y:z), f(z:x), f(x:y)\}$$
$$= \sigma\{n, r, f(y:z)\} + \sigma\{n, r, f(z:x)\} + \sigma\{n, r, f(x:y)\}$$
$$= 3C(n-1, r-2) \ldots\ldots (24a).$$

8a. *Computation of* $\sigma\{n, r, f(y, z), f(z, x), f(x, y)\}$. This may be decomposed into the sum of three parts as follows:

Let $\sigma\{n, r, f(y, z)\}$, $\sigma\{n, r, f(z, x)\}$, $\sigma\{n, r, f(x, y)\}$ be the number of *non-homogeneous ternary* n-ic functions of two variables, *i.e.* of y, z; or of z, x; or of x, y respectively, containing r terms, but not containing any of x, y, or z as a factor.

These functions are evidently *proper ternary* n-ics; the number of these has been computed in the previous paper on the "Number of Ternary n-ics," p. 5, and is denoted for shortness by $N'''(n, r)$.

Thus $\sigma\{n, r, f(y, z)\} = \sigma\{n, r, f(z, x)\}$
$$= \sigma\{n, r, f(x, y)\} = N'''(n, r) \ldots\ldots\ldots (25).$$

Hence, as the sets included in the above three numbers σ are wholly different, these three parts are additive,

therefore $\sigma\{n, r, f(y, z), f(z, x), f(x, y)\}$
$$= \sigma\{n, r, f(y, z)\} + \sigma\{n, r, f(z, x)\} + \sigma\{n, r, f(x, y)\}$$
$$= 3N'''(n, r) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25a).$$

By equation (29) of previous Paper,

$$N^{iv}(n, r) = \{C(T_n, r) - 3C(T_{n-1}, r) + 3C(T_{n-2}, r) - C(T_{n-3}, r)\}$$
$$- 3C(n-1, r-2) \text{ ....(26)}.$$

8b. *Computation of* $\sigma\{n, r, f(x:y:z)\}$. By definition V (Art. 3), this is the number of *homogeneous ternary* n-ic *functions of three variables*, containing r terms, but not containing x, y, or z as factors: these functions are therefore all contained in U_n, and are in fact all *proper ternary* n-ics, so that their number is $N'''(n, r)$, see notation, Art. 8d,

therefore $\sigma\{n, r, f(x:y:z)\} = N'''(n, r)$ ....(27).

9. *Recomposition of* $\sigma(n, r)$. Combining the different parts of $\sigma(n, r)$ from Results (10), (21), (23a), (24a), (25a), (27),

$$\sigma(n, r) = \{3C(\tau_{n-1}, r) - 6C(\tau_{n-2}, r) + 4C(\tau_{n-3}, r) - C(\tau_{n-4}, r)\}$$
$$+ 6C(n-1, r-2) + 4N'''(n, r) \text{ ....(28)}.$$

10. *Final formula for* $N^{iv}(n, r)$. By equations (9a), (7a), (28), (26), the number sought is

$$N^{iv}(n, r) = \{C(\tau_n, r) - 4C(\tau_{n-1}, r) + 6C(\tau_{n-2}, r) - 4C(\tau_{n-3}, r)$$
$$+ C(\tau_{n-4}, r)\} - 4\{C(T_n, r) - 3C(T_{n-1}, r) + 3C(T_{n-2}, r)$$
$$- C(T_{n-3}, r)\} + 6C(n-1, r-2) \text{ .. ...(29)},$$

wherein by equations (4b), (4c),

$$T_n = \tfrac{1}{2}(n+1)(n+2), \text{ \&c.}; \ \tau_n = \tfrac{1}{6}(n+1)(n+2)(n+3), \text{ \&c.}$$
$$\text{.........(30)},$$

and the range of r is easily seen to be

r not < 2, nor $> \tau_n$, *i.e.* not $> \frac{1}{6}(n+1)(n+2)(n+3)$...(31).

Thus the expression for $N^{iv}(n, r)$ is the algebraic sum of ten terms of form C; therefore the greater the value of r, the simpler the general expression for $N^{iv}(n, r)$ becomes by the property of C,

$$C(m, r) = 0, \text{ when } r > m \text{ ....(32)},$$

so that each term C vanishes in turn as r increases through the critical values,

$$r > (n+1); \ T_{n-3}, T_{n-2}, T_{n-1}, T_n;$$
$$\tau_{n-4}, \tau_{n-3}, \tau_{n-2}, \tau_{n-1} \text{ ....(31}a\text{)},$$

and finally

$$N''(n, r) = C(\tau_n, r), \text{ when } n > \tau_{n-1} \text{ ....(33).}$$

[In the previous Paper (p. 6) a Table of the reduced values of $N'''(n, r)$ was given, showing the forms assumed as r increases through the critical values: a similar set of reductions of formula (29) could of course be given for $N''(n, r)$, but it does not seem worth while, as the reductions follow obviously from formula (32)].

11. *Computation of* $N''(n)$. From the definition, Art. 2, the final value sought is

$$N''(n) = \Sigma \{N''(n, r)\} \text{ from } r = 2 \text{ to } r = \tau_n \text{ .....(34).}$$

But, by the Theory of Combinations,

$$\Sigma_{r=2}^{r=m} \{C(m, r)\} = \Sigma_{r=1}^{r=m} \{C(m, r)\} - C(m, 1)$$

$$= (2^m - 1) - m \text{ ....(35}a\text{),}$$

and

$$\Sigma_{r=2}^{r=n+1} \{C(n-1, r-2)\} = \Sigma_{\rho=0}^{\rho=n-1} (n-1, \rho) = 2^{n-1} \text{ ...(35}b\text{),}$$

therefore by Results (29), (34), (35a, b), $N''(n)$ becomes

$$\begin{aligned} N''(n) = & (2^{\tau_n} - 4 \times 2^{\tau_{n-1}} + 6 \times 2^{\tau_{n-2}} - 4 \times 2^{\tau_{n-3}} + 2^{\tau_{n-4}}) \\ & - (\tau_n - 4\tau_{n-1} + 6\tau_{n-2} - 4\tau_{n-3} + \tau_{n-4}) \\ & - 4 \times (2^{T_n} - 3 \times 2^{T_{n-1}} + 3 \times 2^{T_{n-2}} - 2^{T_{n-3}}) \\ & + 4 \times (T_n - 3T_{n-1} + 3T_{n-2} - T_{n-3}) \\ & + 6 \times 2^{n-1}, \end{aligned}$$

and, as the 2nd and 4th lines of the above are both zero, this reduces to

$$N''(n) = (2^{\tau_n} - 4 \times 2^{\tau_{n-1}} + 6 \times 2^{\tau_{n-2}} - 4 \times 2^{\tau_{n-3}} + 2^{\tau_{n-4}})$$

$$- 4 \times (2^{T_n} - 3 \times 2^{T_{n-1}} + 3 \times 2^{T_{n-2}} - 2^{T_{n-3}}) + 6 \times 2^{n-1} \text{ ...(36)}$$

a remarkable formula, in which the law of the coefficients is pretty obvious.

[By substituting for τ and T in terms of n, this can be reduced so as to exhibit $N''(n)$ as an explicit function of n, similar to the final formula for $N'''n$, see equation (36a) of the previous Paper: but the Result is so complicated that it does not seem worth printing].

The following Table shows the value of T_n, τ_n, $N''(n, r)$ and $N''(n)$ for the n-ic quantics defined by $n = 1, 2, 3$. It will be seen how very rapidly $N''(n)$ increases with n; in fact $N''(4)$ is not much less than 2^{35} which runs into 11 figures.

n	T_n	τ_n	Values of $N''(n, r)$ for all values of r.											Value of $N''(n)$
1	3	4	r	1	2	3	4	...	...	...	...	...	...	1
			$N''(n, r)$	0	0	0	1	...	...	...	...	...	...	
2	6	10	r	1	2	3	4	5	6	7	8	9	10	801
			$N''(n, r)$	0	3	42	146	228	206	120	45	10	1	
3	10	20	r	1	2	3	4	5	6	7	8	9	10	1,041,177
			$N''(n, r)$	0	16	444	3,357	13,560	37,092	76,560	125,610	167,880	184,748	
			r	11	12	13	14	15	16	17	18	19	20	
			$N''(n, r)$	167,960	125,970	77,520	38,760	15,504	4,845	1,140	190	20	1	

[To ensure accuracy the values of $N''(n)$ have been computed from the general formula 36, and found to tally with those given by adding the values of $N''(n, r)$. The values of $N''(n, r)$ have also been verified for the case of $n = 2$ when $r = 2, 3, 4$ by actually writing out the quaternary n-ics themselves].