NOTE ON THE MAXIMUM NUMBER OF ARBITRARY POINTS WHICH CAN BE DOUBLE POINTS ON A CURVE, OR SURFACE, OF ANY DEGREE.

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If a curve of the n^{th} degree be given by r double points, and t ordinary points, the usual rule connecting these numbers is $3r + t = \frac{1}{2}\{n(n+3)\}$.

Suppose $n = 2$; then $3r + t = 5$, so we might conclude that a conic could not have two double points chosen arbitrarily. But the square of the line, joining any two points, is a case of a conic with two double points chosen arbitrarily, and an infinite number of double points depending on these two.

Suppose $n = 4$; then $3r + t = 14$, so we might conclude that a quartic could not have five double points. But the square of the conic, through any five points, is a quartic with five double points chosen arbitrarily, and an infinite number of double points depending on these five.

I propose to shew that these are the only cases of exception to the usual rule for plane curves; and that there are, for surfaces of the second and fourth degrees only, similar exceptions to the rule $4r + t = \frac{1}{6}\{n(n^2 + 6n + 11)\}$. Having proved these propositions, it will be easy to deduce some theorems (mostly well-known), regarding certain canonical forms of algebraical curves and surfaces.

To be given that a curve passes through t points is to be given t independent linear conditions; I have now to shew that (with the exceptions stated) to be given m double points, and s ordinary points, is equivalent to $3m + s$ independent linear conditions. Assuming the rule to hold for $m = r - 1$, it may be shewn to hold for $m = r$. Suppose it was only equivalent to $3r + s - 1$ independent conditions. Add a sufficient number of ordinary points, to be also passed through, so as to make

$$3r + t - 1 = \tfrac{1}{2}\{n(n+3)\} \text{ or } 3(r-1) + t = \tfrac{1}{2}\{n(n+3)\} - 2.$$

By what we have assumed, a system of curves having $r - 1$ given double points, and t ordinary points, satisfies $3(r-1) + t$ independent linear conditions; and therefore the system must take the form $\lambda U + \mu V + \nu W = 0$, where U, V, W are particular curves of the system, and λ, μ, ν parameters.

The curve which has r double points, and t ordinary points given, is therefore a particular member of this system.

Now the double points, of such a system, lie on the Jacobian

$$\begin{vmatrix} U_1, & V_1, & W_1 \\ U_2, & V_2, & W_2 \\ U_3, & V_3, & W_3 \end{vmatrix} = 0,$$

and therefore the r_{th} double point, belonging to the particular curve must lie on this locus; but that double point was supposed arbitrary, therefore the Jacobian must vanish identically.

If to every point P on a curve a near point P' also on the curve can be found, such that PP' is not the tangent at P, it can only be because the curve ultimately breaks up into the square of a curve of half its degree. Let now P, P', P'' be any three points very near to one another. One curve of the system can be drawn through P, P', and another through P, P'', and the polar lines of P with regard to these are the tangents PP' and PP'' respectively, which intersect in P. The polar of P, with regard to every other curve, must therefore also pass through P, (since this is the geometrical interpretation of the Jacobian vanishing identically); that is, every other curve must pass through P, which is impossible, P being an arbitrary point. PP' cannot therefore be a tangent, and the curves of the system must break up into curves of degree $\frac{1}{2}n$; that is $\lambda U + \mu V + \nu W = 0$ takes the form $(S + \lambda S')(S + \mu S') = 0$; and any other point may be considered a double point, for that curve $(S + \lambda S')^2 = 0$ which passes through it.

The curve $S + \lambda S' = 0$, which is of degree $\frac{1}{2}n$, contains $r + t$ points arbitrarily chosen; therefore $r + t = \frac{1}{2}\{\frac{1}{2}n(\frac{1}{2}n + 3)\}$, and we had also $3r + t = \frac{1}{2}\{n(n+3)\} + 1$. Of these two equations, it is easily seen that the only solutions are

$$\begin{cases} n = 2, \\ r = 2, \\ t = 0, \end{cases} \quad \text{and} \quad \begin{cases} n = 4, \\ r = 5, \\ t = 0. \end{cases}$$

We have therefore proved by induction that, except when n is 2 or 4, to be given m double points and s ordinary points is equivalent to $3m + s$ independent conditions; and the conditions being independent, it is obvious that they are also linear.

So, to be given that a surface has m double points and s

ordinary points is $4m+s$ independent conditions. Suppose it less; then we can prove, as before, that the Jacobian of U, V, W, T, four particular surfaces of a system, must vanish identically. Let now P, P', P'', P''' be any four near points, and yet such that the angles of the tetrahedron formed by them are finite. Then four surfaces of the system can be described through $PP'P''$, $PP''P'''$, $PP'''P'$, and $P'P''P'''$ respectively; and (neglecting the case of the surfaces being cones with a common vertex, which need not be considered, as it falls under the rule for plane curves) these are typical surfaces of the system. As in the corresponding proof for plane curves, we see that, since these four planes have not a common point, and yet the Jacobians must vanish identically, each surface must break up. That is

$$\lambda U+\mu V+\nu W+\delta T=0$$

must take the form

$$(S+\lambda S'+\mu S'')(S+\nu S'+\delta S'')=0.$$

In the latter form λ, μ, ν, δ are connected by a linear relation $a\lambda+b\mu+c\nu+d\delta=0$, so that there are really only three free parameters. Any other point may be considered a double point for the surface $(S+\lambda S'+\mu S'')^2=0$ which passes through it.

Now $S+\lambda S'+\mu S''=0$ can be described through one arbitrary point only, since $(a+c)\lambda+(b+d)\mu=0$; therefore a surface of degree $\frac{1}{2}n$ can be described through $r+t$ points. We conclude that n must be an even integer, and

$$r+t \overline{\overline{<}} \tfrac{1}{6}\{\tfrac{1}{2}n(\tfrac{1}{4}n^2+3n+11)\};$$

and therefore, since $4r+t>\frac{1}{6}\{n(n^2+6n+11)$, we must have either

$$\begin{cases} n=2, \\ r=3, \\ t=0, \end{cases} \quad \text{or} \quad \begin{cases} n=4, \\ r=9, \\ t=0. \end{cases}$$

If $n=2$, the square of the plane passing through three given points is a quadric with the three points as double points; and the rule giving the number of conditions which can be satisfied is too small by three.

If $n=4$, a quadric can be described through nine points, and the square of this a quartic surface with nine double points; the rule giving too small a number of free conditions by two.

The fact that curves and surfaces of the second degree and of the fourth degree have an abnormal number of double points gives some determinants and matrices which vanish identically. Thus

$$\begin{vmatrix} \alpha_1, & \alpha_2, & 0, & 0, & 0, & 0 \\ \beta_1, & \beta_2, & \alpha_1, & \alpha_2, & 0, & 0 \\ \gamma_1, & \gamma_2, & 0, & 0, & \alpha_1, & \alpha_2 \\ 0, & 0, & \beta_1, & \beta_2, & 0, & 0 \\ 0, & 0, & \gamma_1, & \gamma_2, & \beta_1, & \beta_2 \\ 0, & 0, & 0, & 0, & \gamma_1, & \gamma_2 \end{vmatrix} \equiv 0,$$

since a conic can have two double points; and

$$\left\| \begin{array}{cccccccccccc} \alpha_1, & \alpha_2, & \alpha_3, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\ \beta_1, & \beta_2, & \beta_3, & \alpha_1, & \alpha_2, & \alpha_3, & 0, & 0, & 0, & 0, & 0, & 0 \\ \gamma_1, & \gamma_2, & \gamma_3, & 0, & 0, & 0, & \alpha_1, & \alpha_2, & \alpha_3, & 0, & 0, & 0 \\ \delta_1, & \delta_2, & \delta_3, & 0, & 0, & 0, & 0, & 0, & 0, & \alpha_1, & \alpha_2, & \alpha_3 \\ 0, & 0, & 0, & \beta_1, & \beta_2, & \beta_3, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & \gamma_1, & \gamma_2, & \gamma_3, & \beta_1, & \beta_2, & \beta_3, & 0, & 0, & 0 \\ 0, & 0, & 0, & \delta_1, & \delta_2, & \delta_3, & 0, & 0, & 0, & \beta_1, & \beta_2, & \beta_3 \\ 0, & 0, & 0, & 0, & 0, & 0, & \gamma_1, & \gamma_2, & \gamma_3, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & \delta_1, & \delta_2, & \delta_3, & \gamma_1, & \gamma_2, & \gamma_3 \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \delta_1, & \delta_2, & \delta_3 \end{array} \right\| \equiv 0,$$

since a quadric can have three double points. The results when $n = 4$ are also simple, but take up too much space to write down.

It is noticed in Salmon's *Higher Plane Curves* (3rd edition, Art. 45), that if a curve of the sixth degree have its maximum number of arbitrary double points, that is nine, it must be the square of a cubic. This is the only case excepting those already discussed where a curve with its maximum number of arbitrary double points can degenerate.

Suppose a curve of degree $m + n$ breaks up into an m^{ic} and n^{ic}; and let its double points be made up of t intersections of the m^{ic} and n^{ic} ; p double points on the m^{ic} and q on the n^{ic}; its ordinary points by t_1 points on the m^{ic} and t_2 points on the n^{ic}.

Then we have

$$t+3p+t_1=\tfrac{1}{2}\{m(m+3)\},$$

$$t+3q+t_2=\tfrac{1}{2}\{n(n+3)\},$$

$$3p+3q+3t+t_1+t_2=\tfrac{1}{2}\{(m+n)(m+n+3)\};$$

therefore $t=mn.$

But the number of arbitrary points of intersection of an m^{ic} and n^{ic} if $(m \not< n)$ cannot be greater than $\frac{1}{2}\{m(m+3)\}$; therefore $mn \not> \frac{1}{2}\{m(m+3)\}$, that is $2n \not> m+3$, of which the only solution is $m=n=3$.

If an exceptional case were to arise through m being equal to four, the m^{ic} would be determined completely, and we should have

$$p=5,\ t=t_1=0,\ 3q+t_2=\tfrac{1}{2}\{n(n+3)\},$$

$$15+3q+t_2=\tfrac{1}{2}\{(n+4)(n+7)\},$$

which gives $4n=1$. A similar absurdity would result from the supposition $m=2$.

I have now to prove that any n^{ic} can be written

$$U\equiv(\alpha_1x+\beta_1y+\gamma_1z)^n+(\alpha_2x+\beta_2y+\gamma_2z)^n+\dots$$
$$+(\alpha_rx+\beta_ry+\gamma_rz)^n,$$

(where $3r=\frac{1}{2}\{(n+1)(n+2)\}+t$, and t is 0, or 2) with t degrees of freedom, except when n is 2, or 4.

First consider $t=0$; that is, when n is not a multiple of 3.

The only cases of exception to the theorem must be when

$$\Sigma\alpha^n,\ \Sigma\alpha^{n-1}\beta,\ \Sigma\alpha^{n-1}\gamma,\ \dots,\ \Sigma\beta^n,\ \Sigma\beta^{n-1}\gamma,\ \dots,\ \Sigma\gamma^n$$

are not independent functions of the $\frac{1}{2}\{(n+1)(n+2)\}$ variables

$$\alpha_1,\beta_1,\gamma_1,\alpha_2,\beta_2,\gamma_2,\dots,\alpha_r,\beta_r,\gamma_r;$$

for any relation between these functions would be a relation between the coefficients of the given n^{ic}.

The condition for a relationship is the vanishing of the Jacobian of the functions.

Forming the Jacobian we get the determinant, which expresses that α, β, γ, and the other $r-1$ points should be arbitrary double points on some n^{ic}.

Supposing this determinant were to vanish identically, we should have r double points on an n^{ic} where $3r = \frac{1}{2}\{n(n+3)\}+1$; and this we saw was impossible unless n were 2, or 4.

The only cases of exception are then when n is 2, or 4.

Thus a conic cannot be thrown into the form of the sum of two squares, though we have apparently five constants at our disposal; nor can a quartic be thrown into the form of the sum of five fourth powers, which is Clebsch's theorem.

Consider now $t=2$; that is, n is a multiple of three.

Choose for β_r and γ_r arbitrarily; then there will be left $3r-2$ variables, that is, $\frac{1}{2}\{(n+1)(n+2)\}$.

The vanishing of the Jacobian would express that an n^{ic} could have $r-1$ double points, and one ordinary point arbitrarily selected, where $3(r-1)+1 = \frac{1}{2}\{(n+1)(n+2)\}$; and this we saw was impossible.

It is not difficult to see that in the cases of exception a conic could be reduced to the sum of three squares, with three degrees of freedom, providing we do not choose α_3, β_3, and γ_3, arbitrarily; that is, we might take α_1, β_3, and γ_3, at choice, but not α_1, β_1, and γ_1, or α_3, β_3, and γ_3.

Now also a quartic can be reduced to the sum of six fourth powers, with three degrees of freedom, subject to the same restrictions.

In the same manner it is deduced that any surface of the n^{th} degree may be written

$$U \equiv (\alpha_1 x + \beta_1 y + \gamma_1 z + \delta_1 w)^n + \ldots + (\alpha_r x + \beta_r y + \gamma_r z + \delta_r w)^n,$$

(where $4r = \frac{1}{6}\{(n+1)(n+2)(n+3)\}+t$; t being 0, 1, 2, or 3); and the degrees of freedom with which this may be done is t; but if n is 2 or 4 the theorem is not true.

Thus a quadric cannot be thrown into the form of the sum of three squares, though there are two more constants at our disposal than appear sufficient; nor a quartic into the form of the sum of nine fourth powers, though there is one more constant than appears sufficient.

A cubic surface can, however, be expressed as the sum of five cubes; this is Sylvester's theorem.

A surface of the fifth degree can be expressed as the sum of fourteen fifth powers, and so in general.

In the exceptional cases, and subject to the same restrictions, as in the case of plane curves, a quadric can be expressed as the sum of four squares, with six degrees of freedom; and a quartic surface as the sum of ten fourth powers with five degrees of freedom.

A plane quartic can, we have seen, be expressed as the sum of six fourth powers.

Let $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \ldots, \alpha_6, \beta, \gamma_6$ be the coordinates of the lines, forming the hexagon of reference; then the invariant, called B in Salmon's *Higher Plane Curves* is the square of the determinant

$$\begin{vmatrix} \alpha_1^2, & \alpha_2^2, & \alpha_3^2, & \alpha_4^2, & \alpha_5^2, & \alpha_6^2 \\ \alpha_1\beta_1, & \alpha_2\beta_2, & \ldots & \ldots & \ldots & \ldots \\ \alpha_1\gamma_1, & \alpha_2\gamma_2, & \ldots & \ldots & \ldots & \ldots \\ \beta_1^2, & \beta_2^2, & \ldots & \ldots & \ldots & \ldots \\ \beta_1\gamma_1, & \beta_2\gamma_2, & \ldots & \ldots & \ldots & \ldots \\ \gamma_1^2, & \gamma_2^2, & \ldots & \ldots & \ldots & \ldots \end{vmatrix}.$$

This invariant (the catalecticant of the quartic) vanishes then, not only if the quartic can be expressed as the sum of five fourth powers but, also if the sides of the hexagon of reference touch a conic, and this includes the first theorem.

The corresponding theorem for a quartic surface is that its catalecticant vanishes, if the faces of the dekahedron of reference touch a quadric.

The curve of the sixth degree can be expressed as the sum of ten sixth powers; and if the sides of this decagon touch a curve of the third class, the catalecticant of the sextic vanishes.

ON THE APPLICATION OF ABEL'S THEOREM TO ELLIPTIC INTEGRALS OF THE FIRST KIND.

By *W. Burnside.*

TAKE as the fixed curve for an application of Abel's theorem the quartic

$$y^2 = \frac{ax^2+2bx+c}{Ax^2+2Bx+C} = \frac{N_x}{D_x} \text{ say } \ldots\ldots\ldots\ldots\text{(i)},$$

and for the variable curve the hyperbola

$$y = \frac{mx+n}{m'x+n'}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

Then if $N_xD_x = X$, a general quartic function,

$$\sum_1^4 \frac{dx_r}{\sqrt{X_r}} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)},$$

where x_1, x_2, x_3, x_4 are the abscissæ of the points of intersection of (i) and (ii), when the constants m, n, m', n' vary in any way.