But $$(1+t)^{2n} = 1 + 2nt + \dots + \frac{2n!}{(2n-k)!\,k!} t^k + \dots\dots,$$

$$(1+t^{-1})^n = 1 + nt^{-1} + \dots + \frac{n!}{(n-k)!\,k!} t^{-k} + \dots\dots.$$

Multiplying these two series together, we see that

$$\Sigma_{k=0}^{k=n} \frac{2n!}{(2n-k)!\,k!} \times \frac{n!}{(n-k)!\,k!} = \text{absolute term in } \frac{(1+t)^{3n}}{t^n}$$

$$= \frac{3n!}{2n!\,n!}.$$

Therefore

$$1 - a_1^{\,3} + a_2^{\,3} - \dots = (-1)^n \frac{2n}{n!\,n!} \times \frac{3n!}{2n!\,n!}$$

$$= (-1)^n \frac{3n!}{(n!)^3}.$$

NOTE ON THE SIMULTANEOUS TRANSFORMATION OF TWO QUADRATIC FUNCTIONS.

By *J. E. Campbell*, Hertford College, Oxford.

If two quadratics in n variables $x_1, x_2, x_3, \dots x_n$,

$$u \equiv a_{11}x_1^{\,2} + a_{22}x_2^{\,2} + \dots + 2a_{12}x_1x_2 + \dots,$$

and $$v \equiv b_{11}x_1^{\,2} + b_{22}x_2^{\,2} + \dots + 2b_{12}x_1x_2 + \dots,$$

be transformed by the linear substitution,

$$x_1 \equiv l_1X_1 + l_2X_2 + \dots,$$
$$x_2 \equiv m_1X_1 + m_2X_2 + \dots,$$
$$\&c., \qquad \&c.,$$

they take the forms

$$U \equiv A_{11}X_1^{\,2} + A_{22}X_2^{\,2} + \dots + 2A_{12}X_1X_2 + \dots,$$
$$V \equiv B_{11}X_1^{\,2} + B_{22}X_2^{\,2} + \dots + 2B_{12}X_1X_2 + \dots.$$

It is well-known that the necessary conditions that $A_{12}, A_{13}, \dots, B_{12}, B_{13}, \dots$, should all vanish is that $l_1 : m_1 : n_1 : \dots$ should be proportional to the first minors of

$$\begin{vmatrix} a_{11}+\lambda_1 b_{11}, & a_{12}+\lambda_1 b_{12}, & a_{13}+\lambda b_{13}, \dots, \\ a_{12}+\lambda_1 b_{12}, & a_{22}+\lambda_1 b_{22}, & \dots \quad , \dots, \\ a_{13}+\lambda_1 b_{13}, & \dots \quad , & \dots \quad , \dots, \\ \dots & \dots \quad , & \dots \quad , \dots, \end{vmatrix} = 0,$$

www.rcin.org.pl

where λ_1 is a root of the above determinantal equation; and that

$$l_2 : m_2 : n_2 : \ldots,$$
$$l_3 : m_3 : n_3 : \ldots,$$
$$\&c., \quad \&c.,$$

should be determined by similar rules from $\lambda_2, \lambda_3, \ldots$.

Conversely, it may be easily shown that for such a transformation

$$A_{12} + \lambda_1 B_{12} = 0,$$
$$A_{12} + \lambda_2 B_{12} = 0,$$
$$\&c., \quad \&c.,$$

so that, provided $\lambda_1, \lambda_2, \ldots$, are distinct, $A_{12} = B_{12} = 0$, &c., or the conditions are sufficient if Lagrange's determinant has not equal roots.

To prove this, consider two symmetrical determinants of the third order (though the proof is general) which each vanish;

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0 \text{ and } \begin{vmatrix} a', & h', & g' \\ h', & b', & f' \\ g', & f', & c' \end{vmatrix} = 0,$$

then if $A, H, G, \ldots, A', H', G', \ldots$ be their respective first minors,

$$A(aA' + hH' + gG') + H(hA' + bH' + fG') \\ + G(gA' + fH' + cG') = 0,$$

and by interchanging a and a', b and b', &c., we get another similar equation. They are proved at once since

$$aA + hH + gG = 0,$$
$$hA + bH + fG = 0,$$
$$gA + fH + cG = 0,$$

and similar equations hold for the dotted letters.

It follows that if $l_1 : m_1 : n_1, \ldots,$

$$l_2 : m_2 : n_2, \ldots,$$

have the values found for them,

$$l_1[(a_{11} + \lambda_1 b_{12}) l_2 + (a_{12} + \lambda_1 b_{12}) m_2 + \ldots] \\ + m_1[(a_{12} + \lambda_1 b_{12}) l_2 + (a_{22} + \lambda_1 b_{22}) m_2 + \ldots] \\ + \ldots = 0,$$

www.rcin.org.pl

that is $A_{12} + \lambda_1 B_{12} = 0$, and the similar equation gives $A_{12} + \lambda_2 B_{12} = 0$.

Suppose now that two roots of the equation, say λ_1 and λ_2, are equal. The necessary conditions then give

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \ldots,$$

unless all the first minors vanish. That is

$$\begin{aligned} x_1 &\equiv l_1 (X_1 + X_2) + l_3 X_3 + \ldots, \\ x_2 &\equiv m_1 (X_1 + X_2) + m_3 X_3 + \ldots, \\ &\&c., \qquad\qquad \&c., \end{aligned}$$

or x_1, x_2, ..., are now expressible in terms of $m-1$ new variables, which is impossible since x_1, x_2, ..., are independent. We conclude then that the reduction when Lagrange's determinant has a pair of equal roots is impossible unless all the first minors vanish.

The following are simple examples of this failure.

If
$$\begin{aligned} u &\equiv ax^2 + 2hxy + by^2, \\ v &\equiv a'x^2 + 2h'xy + b'y^2, \end{aligned}$$

have a single common factor, then

$$\begin{vmatrix} a + \lambda a', & h + \lambda h' \\ h + \lambda h', & b + \lambda b' \end{vmatrix} = 0$$

has equal roots, and the equations cannot be reduced to the normal forms $u \equiv x^2 + y^2$, $v \equiv \alpha x^2 + \beta y^2$.

If two conics

$$\begin{aligned} u &\equiv (a\,, b\,, c\,, f\,, g\,, h\,)\,(x, y, z)^2 = 0, \\ v &\equiv (a', b', c', f', g', h')\,(x, y, z)^2 = 0, \end{aligned}$$

have single contact, the discriminant of $u + kv$ has equal roots, and the conics cannot be reduced to the normal forms

$$u \equiv x^2 + y^2 + z^2 = 0, \;\; v \equiv ax^2 + by^2 + cz^2 = 0.$$

So also, if two quadrics have single contact they cannot be reduced to the normal forms.

This failure explains some anomalies. Two conics can, in general, be written $x^2 + y^2 + z^2 = 0$, $ax^2 + by^2 + cz^2 = 0$, and if $a = b$ the conics have double contact, apparently one condition instead of two.

www.rcin.org.pl

Two quadrics can be written

$$x^2 + y^2 + z^2 + w^2 = 0,$$
$$ax^2 + by^2 + cz^2 + dw^2 = 0.$$

If $a = b$ they interest in planes or have double contact which requires two conditions.

It is only a particular case of the last that any quadric may be reduced to the form $ax^2 + by^2 + cz^2 + dw^2 = 0$; if $a = b = 0$ it reduces to a pair of planes. Two conditions here take the place of three.

These all arise from the reduction failing unless a further condition holds; assuming which implicitly we seem to arrive at less than the proper number of conditions.

Suppose now that all the first minors do vanish for λ_1, the reduction to the normal forms will again be possible. The quantities $l_1 : m_1 : n_1 : \ldots$ have now only to satisfy $n-2$ linear independent equations. They may therefore each be expressed linearly in terms of one variable θ_1; and similarly $l_2 : m_2 : n_2 : \ldots$ in terms of another variable θ_2. We have also, as before, $A_{12} + \lambda_1 B_{12} = 0$, and if we connect θ_1, θ_2 by putting $A_{12} = 0$, we get $B_{12} = 0$ also, or the reduction is now possible and with one degree of freedom.

Examples. If two conics

$$u \equiv (a, b, c, f, g, h)(x, y, z)^2 = 0,$$
$$v \equiv (a', b', c', f', g', h')(x, y, z)^2 = 0,$$

have double contact, they can be thrown into the forms

$$x^2 + y^2 + z^2 = 0, \quad x^2 + y^2 + cz^2 = 0.$$

The first minors of discriminant of $u + kv$ must then all vanish, and this gives the ordinary form of conditions that two conics have double contact.

If two quadrics $u = 0$, $v = 0$ intersect in planes, they can be reduced to the forms

$$u \equiv x^2 + y^2 + z^2 + w^2 = 0,$$
$$v \equiv a(x^2 + y^2) + cz^2 + dw^2 = 0,$$

all the first minors of discriminant of $u + kv$ must then vanish, or, if the first minor be $A + k\alpha + k^2\alpha' + k^3 A'$, &c., &c.,

$$\left\| \begin{matrix} A, & B, & C, & D, & F, & G, & H, & L, & M, & N \\ \alpha, & \ldots & & & & & & & & \ldots \\ \alpha', & \ldots & & & & & & & & \ldots \\ A', & \ldots & & & & & & & & \ldots \end{matrix} \right\| = 0,$$

gives the conditions that a pair of quadrics should have double contact.

www.rcin.org.pl

Similar conditions to these are obtained, if, in general, $u+kv$ can be reduced to the sum of $n-2$ squares.

If three roots of Lagrange's determinant are equal, the reduction becomes impossible again. For, let $\lambda_1=\lambda_2=\lambda_3$, we have, as before,

$$A_{12}+\lambda_1 B_{12}=0,$$

$$A_{13}+\lambda_1 B_{13}=0,$$

$$A_{23}+\lambda_1 B_{23}=0,$$

and

$$l_1 : m_1 : n_1 : \ldots,$$

$$l_2 : m_2 : n_2 : \ldots,$$

$$l_3 : m_3 : n_3 : \ldots,$$

have each been determined with one degree of freedom, θ_1, θ_2, θ_3 determining these by the equations

$$A_{12}=A_{23}=A_{31}=0.$$

The ratios $l_1 : m_1 : n_1 : \ldots$ will now be equal to the ratios

$$l_2 : m_2 : n_2 : \ldots,$$

$$l_3 : m_3 : n_3 : \ldots,$$

and for the same reason as before this is impossible.

If, however, all the second minors vanish the reduction will again be possible, for

$$l_1 : m_1 : n_1 : \ldots,$$

$$l_2 : m_2 : n_2 : \ldots,$$

$$l_3 : m_3 : n_3 : \ldots,$$

have now only to satisfy $n-3$ independent equations; they can therefore each be determined linearly in terms of θ_1, ϕ_1; θ_2, ϕ_2, and θ_3, ϕ_3 respectively; and requiring these to satisfy the three equations $A_{12}=A_{23}=A_{31}=0$, we have also $B_{12}=B_{23}=B_{31}=0$, and

$$l_1 : m_1 : n_1 : \ldots,$$

$$l_2 : m_2 : n_2 : \ldots,$$

$$l_3 : m_3 : n_3 : \ldots,$$

are determined with three degrees of freedom.

We thus get for example the conditions that two quadrics should have plane contact.

www.rcin.org.pl

More generally, we may similarly prove that if r roots of Lagrange's determinant are equal, the reduction of the quadratics to the normal forms is impossible unless all the $(r-1)^{\text{th}}$ minors vanish for that equal root; but in case they do, the reduction is possible with $\frac{1}{2}r(r-1)$ degrees of freedom.

It is only a particular case of the theorem that if any quadratic in n variables can be reduced to the sum of $n-r$ squares, all the $(r-1)^{\text{th}}$ minors of its discriminant must vanish.

Thus the conditions that the quadratic should break into linear factors are obtained.

The number of independent conditions that a quadratic should be reduced to the sum of $n-r$ squares is $\frac{1}{2}r(r+1)$; this therefore expresses the number of conditions to which the vanishing of $(r-1)^{\text{th}}$ minors is equivalent.

If one of the quadratics be limited by the conditions of being essentially positive or negative, we know that if Lagrange's determinant have r equal roots, all the first minors have $r-1$ equal roots. In this case then the reduction must always be possible, for the necessary vanishing of the minors is assured.

Thus in the oscillations of a system about the position of equilibrium, the equality of two or more of the periods does not prevent us from referring to principal coordinates, but leaves freedom in the choice of the latter.

So in reducing the equation of the quadric to its principal axes, Lagrange's determinant whose first minors give the direction cosines of the axes will be

$$\begin{vmatrix} a+\lambda, & h\ , & g \\ h\ , & b+\lambda, & f \\ g\ , & f\ , & c+\lambda \end{vmatrix} = 0,$$

and this does not break down for the case of equal roots, but gives an indeterminate pair of principal axes; also since the first minors all vanish, we get the ordinary conditions for a quadric of revolution

$$a-\frac{gh}{f}=b-\frac{fh}{g}=c-\frac{fg}{h}.$$

A quadric can then by orthogonal transformation be reduced to the form $ax^2+by^2+cz^2=1$; if $a=b$ it is of revolution, apparently one condition; but, as in the general case, the other condition required is implicitly given in the fact that the vanishing of the minors is necessary for the reduction if the discriminating cubic has equal roots.

www.rcin.org.pl